

Honors Geometry Study Guide

*Core “hard” concepts of an advanced high school geometry course,
for students self-studying beyond the Khan Academy curriculum*

- 1.** Formal Proof Writing
- 2.** Triangle Centers and Advanced Triangle Relationships
- 3.** Similarity Beyond the Basics
- 4.** Circles — The Deep End
- 5.** Trigonometry Extensions
- 6.** Transformations as a Formal System
- 7.** Three-Dimensional Geometry
- 8.** Compass-and-Straightedge Constructions
- 9.** Coordinate Geometry Synthesis
- 1** Honors-Level Extras and Enrichment
- 0.**

Answer Key

How to use this guide: read each topic summary, then attempt every practice problem on paper before checking the Answer Key at the back. Problems are numbered by section (e.g., 4.3 is the third problem in Section 4).

Section 1: Formal Proof Writing

The single biggest difference between Khan Academy geometry and an honors class: honors students must construct complete proofs from scratch. Master the logical machinery here before anything else — it underlies every other section.

Proof formats: two-column, paragraph, and flow proofs

A two-column proof lists numbered statements on the left and a justification (given, definition, postulate, or previously proven theorem) for each on the right. A paragraph proof presents the same logical chain in prose; a flow proof shows it as a diagram of boxes and arrows. Every step must be justified — no step may rely on how a figure looks. Honors classes expect you to choose the givens, plan the chain of deductions, and write the entire proof independently.

Practice

- 1.1** Write a two-column proof: If two angles are vertical angles, then they are congruent. (Given: $\angle 1$ and $\angle 2$ are vertical angles formed by intersecting lines, with $\angle 3$ adjacent to both.)
- 1.2** Prove in paragraph form: If $\angle A$ and $\angle B$ are both supplementary to $\angle C$, then $\angle A \cong \angle B$.
- 1.3** In $\triangle ABC$, D is the midpoint of AB and E is the midpoint of AC , and DE is drawn. Given only that $AB \cong AC$, prove $\triangle ADE$ is isosceles.

Indirect proof (proof by contradiction)

To prove a statement indirectly, temporarily assume its negation is true, then reason logically until you reach a contradiction of a known fact, a given, or a theorem. Since the assumption led to an impossibility, the assumption is false and the original statement must be true. The critical skill is stating the negation correctly: the negation of “at most one” is “at least two,” and the negation of “all” is “at least one is not.”

Practice

- 1.4** Prove indirectly: A triangle can have at most one right angle.
- 1.5** Prove indirectly: If two lines cut by a transversal form alternate interior angles that are NOT congruent, then the lines are not parallel.
- 1.6** What is the correct first line (the assumption) of an indirect proof of: “In $\triangle ABC$, if $AB \neq AC$, then $\angle B \neq \angle C$ ”?

Coordinate proofs

A coordinate proof places a figure on the coordinate plane and uses the distance formula, midpoint formula, and slopes to establish geometric facts: equal slopes show parallel lines, slopes with product -1 show perpendicular lines, and the distance formula shows congruent segments. The art is in tool selection — knowing that “prove it’s a rhombus” means “show all four sides have equal length,” while “prove it’s a rectangle” means “show adjacent sides are perpendicular (or diagonals are congruent).”

Practice

- 1.7** Quadrilateral $ABCD$ has vertices $A(0, 0)$, $B(4, 2)$, $C(2, 6)$, $D(-2, 4)$. Prove $ABCD$ is a square.
- 1.8** $\triangle PQR$ has vertices $P(0, 0)$, $Q(6, 0)$, $R(3, 5)$. Use coordinates to prove $\triangle PQR$ is isosceles.
- 1.9** Which is the most efficient coordinate strategy to prove that the diagonals of a parallelogram bisect each other?

Logic: converse, inverse, contrapositive, and biconditionals

For a conditional “if p , then q ”: the converse is “if q , then p ,” the inverse is “if not p , then not q ,” and the contrapositive is “if not q , then not p .” A statement and its contrapositive are logically equivalent; a statement and its converse are NOT. A biconditional (“ p if and only if q ”) is true exactly when both the statement and its converse are true — which is why definitions can be written as biconditionals but most theorems cannot. Honors classes also stress the axiomatic hierarchy: undefined terms (point, line, plane) $\rightarrow$ definitions $\rightarrow$ postulates (accepted without proof) $\rightarrow$ theorems (proved).

Practice

1.10 Statement: “If a quadrilateral is a square, then it has four right angles.” Write the converse, inverse, and contrapositive, and state the truth value of each.

1.11 Can “Two angles are vertical angles if and only if they are congruent” be written as a true biconditional? Explain.

1.12 Classify each as undefined term, definition, postulate, or theorem: (a) point; (b) “through any two points there is exactly one line”; (c) “a midpoint divides a segment into two congruent segments”; (d) “base angles of an isosceles triangle are congruent.”

Section 2: Triangle Centers and Advanced Triangle Relationships

Every triangle has four classical “centers,” each with its own construction and its own special property. Honors courses expect you to know which center solves which problem, plus the inequality theorems Khan touches only lightly.

The four classical centers

Centroid: intersection of the medians; it is the balance point and divides each median in a 2:1 ratio measured from the vertex. **Circumcenter:** intersection of the perpendicular bisectors of the sides; it is equidistant from the three VERTICES and is the center of the circumscribed circle (it lies outside an obtuse triangle, on the hypotenuse midpoint of a right triangle). **Incenter:** intersection of the angle bisectors; equidistant from the three SIDES; center of the inscribed circle; always inside the triangle. **Orthocenter:** intersection of the three altitudes (also outside an obtuse triangle).

Practice

- 2.1** A city wants to build a hospital equidistant from three towns, and separately a fountain equidistant from the three straight roads joining the towns. Which triangle center locates each?
- 2.2** In $\triangle ABC$, median AM has length 18 and centroid G lies on it. Find AG and GM .
- 2.3** For a right triangle, where exactly are the circumcenter and the orthocenter located?

The Euler line

In every non-equilateral triangle, the orthocenter (H), centroid (G), and circumcenter (O) are collinear; the line through them is the Euler line. The centroid always lies between the other two and divides the segment so that $HG = 2 \cdot GO$ — i.e., G is one-third of the way from O to H . The incenter joins the party only when the triangle is isosceles (and in an equilateral triangle all four centers coincide).

Practice

- 2.4** A triangle has circumcenter $O(0, 0)$ and orthocenter $H(6, 3)$. Find its centroid.
- 2.5** Which of the four classical centers is generally NOT on the Euler line, and when does it join the line?

Triangle inequalities and the Hinge Theorem

Triangle Inequality: each side must be shorter than the sum of the other two, so sides a and b can enclose a third side x only when $|a - b| < x < a + b$. **Side-angle relationship:** in any triangle, the longer side lies opposite the larger angle and vice versa. **Hinge Theorem (SAS Inequality):** if two triangles have two pairs of congruent sides, the triangle with the LARGER included angle has the LONGER third side — and its converse compares angles from sides.

Practice

- 2.6** Two sides of a triangle measure 8 and 13. What are the possible lengths of the third side?
- 2.7** In $\triangle ABC$ and $\triangle DEF$, $AB = DE$ and $BC = EF$, with $m\angle B = 50^\circ$ and $m\angle E = 70^\circ$. Compare AC and DF , and name the theorem.
- 2.8** In $\triangle XYZ$, $XY = 9$, $YZ = 12$, $XZ = 7$. List the angles from smallest to largest.

The Midsegment Theorem

A midsegment joins the midpoints of two sides of a triangle. The Midsegment Theorem states it is parallel to the third side and exactly half its length. The three midsegments form the medial triangle, whose perimeter is half the original's and whose area is one-quarter of the original's. The theorem is proved either with coordinates or with a clever SAS-similarity argument — honors classes often ask for the proof itself.

Practice

2.9 In $\triangle ABC$, D and E are midpoints of AB and AC. If $BC = 14$ and $m\angle ABC = 63^\circ$, find DE and $m\angle ADE$.

2.10 The medial triangle of $\triangle PQR$ has perimeter 22. What is the perimeter of $\triangle PQR$, and what fraction of $\triangle PQR$'s area does the medial triangle occupy?

2.11 Sketch a coordinate proof of the Midsegment Theorem for the side joining midpoints of AB and AC, with $A(0,0)$, $B(2b, 0)$, $C(2c, 2d)$.

Section 3: Similarity Beyond the Basics

Khan covers basic similar-triangle ratios well. Honors classes push into the geometric-mean relationships hidden inside right triangles and the proportionality theorems that follow from parallel lines and angle bisectors.

Geometric mean relationships in right triangles

When the altitude is drawn to the hypotenuse of a right triangle, it creates three similar triangles (each similar to the original). Two families of “geometric mean” relations follow. Altitude rule: the altitude is the geometric mean of the two hypotenuse segments, $h = \sqrt{m \cdot n}$. Leg rule: each leg is the geometric mean of the whole hypotenuse and the segment adjacent to that leg, $\text{leg} = \sqrt{c \cdot m}$. These identities are the engine behind many honors problems and reappear in the proof of the Pythagorean theorem by similarity.

Practice

- 3.1** The altitude to the hypotenuse divides the hypotenuse into segments of length 4 and 9. Find the altitude.
- 3.2** Using the same triangle (hypotenuse segments 4 and 9), find the length of the leg adjacent to the segment of length 4.
- 3.3** In right $\triangle ABC$ with the right angle at C, altitude CD meets hypotenuse AB at D. Name the three similar triangles in correct correspondence order.

Side-Splitter Theorem and the Angle Bisector Theorem

Side-Splitter (Triangle Proportionality) Theorem: a line parallel to one side of a triangle that intersects the other two sides divides them proportionally — and conversely, proportional division implies parallelism. Extension: three or more parallel lines cut any two transversals proportionally. Angle Bisector Theorem: an angle bisector of a triangle divides the opposite side into segments proportional to the two adjacent sides ($BD/DC = AB/AC$). These are the workhorse tools for “find the missing length” problems that don’t involve a full pair of similar triangles.

Practice

- 3.4** In $\triangle ABC$, $DE \parallel BC$ with D on AB and E on AC. $AD = 6$, $DB = 4$, $AE = 9$. Find EC.
- 3.5** In $\triangle ABC$, $AB = 8$, $AC = 12$, and $BC = 15$. The bisector of $\angle A$ meets BC at D. Find BD and DC.
- 3.6** Three parallel lines cut one transversal into segments of 5 and 8. If the corresponding first segment of a second transversal is 7.5, find the second segment.

Proving similarity rigorously (AA, SAS~, SSS~)

Honors classes require you to PROVE triangles similar before using them, citing AA (two pairs of congruent angles), SAS~ (two pairs of proportional sides with the included angles congruent), or SSS~ (three pairs of proportional sides). Classic setups include overlapping triangles sharing an angle, vertical-angle “bowtie” configurations with parallel lines, and the right-triangle altitude configuration. After proving similarity, the ratio of areas is the SQUARE of the ratio of corresponding sides — a favorite honors twist.

Practice

- 3.7** $AB \parallel DE$, and segments AE and BD intersect at C (a “bowtie”). Prove $\triangle ABC \sim \triangle EDC$ and find DE if $AB = 10$, $AC = 8$, $CE = 12$.

3.8 Two similar triangles have corresponding sides in ratio 3:5. The smaller has area 27. Find the larger triangle's area.

3.9 In $\triangle ABC$, P lies on AB and Q lies on AC with $AP = 4$, $AB = 10$, $AQ = 6$, $AC = 15$. Is $PQ \parallel BC$? Justify.

Section 4: Circles — The Deep End

This is the densest “new material” chapter in most honors courses, and the area Khan students most often find under-drilled. The angle and segment theorems here all trace back to one result: the Inscribed Angle Theorem.

Inscribed Angle Theorem and its corollaries

An inscribed angle measures half its intercepted arc (a central angle equals its full arc). Corollaries: (1) inscribed angles intercepting the same arc are congruent; (2) an angle inscribed in a semicircle is a right angle (Thales’ theorem); (3) opposite angles of a cyclic quadrilateral (one inscribed in a circle) are supplementary. These four facts unlock most circle-angle problems, and honors exams love chaining several together in one figure.

Practice

- 4.1** Arc AB measures 100° . Find the measure of central angle $\angle AOB$ and inscribed angle $\angle ACB$ (C on the major arc).
- 4.2** Cyclic quadrilateral ABCD has $m\angle A = x + 10$ and $m\angle C = 2x - 40$. Find x and both angles.
- 4.3** AB is a diameter of a circle and C is any other point on the circle. If $m\angle CAB = 37^\circ$, find $m\angle ABC$.

Angles formed by chords, secants, and tangents

Three cases, organized by where the vertex sits. Vertex ON the circle (tangent-chord or inscribed): angle = $\frac{1}{2}$ (intercepted arc). Vertex INSIDE the circle (two chords): angle = $\frac{1}{2}$ (sum of the two vertical intercepted arcs). Vertex OUTSIDE the circle (two secants, two tangents, or secant-tangent): angle = $\frac{1}{2}$ (difference of the far and near arcs). Memorize “on: half; inside: half the sum; outside: half the difference,” then practice until identifying the case is automatic.

Practice

- 4.4** Two chords intersect inside a circle, forming an angle that intercepts arcs of 80° and 40° . Find the angle.
- 4.5** Two secants from an external point intercept arcs of 120° and 40° . Find the angle at the external point.
- 4.6** A tangent and a chord meet at a point of tangency, and the chord cuts off an arc of 150° on one side. Find the tangent-chord angle on that side.
- 4.7** Two tangents are drawn from an external point to a circle, and the angle between them is 40° . Find the measures of the two arcs they determine.

Power of a point: segment relationships

Three product rules for segments through a circle. Chord-chord: if two chords intersect inside, the products of their segments are equal ($a \cdot b = c \cdot d$). Secant-secant: from an external point, (whole secant)·(external part) is the same for both secants. Secant-tangent: the tangent length squared equals (whole secant)·(external part). All three are really one statement — the “power of the point” — proved by spotting similar triangles in the figure, which honors classes will make you do.

Practice

- 4.8** Chords AB and CD intersect at P inside a circle. $AP = 3$, $PB = 8$, $CP = 4$. Find PD.

4.9 From external point P, a tangent has length t and a secant has external segment 4 and total length 9. Find t .

4.10 From external point P, two secants are drawn. One has external segment 5 and chord portion 7; the other has external segment 6. Find the second secant's chord portion.

Equations of circles and completing the square

A circle with center (h, k) and radius r has equation $(x - h)^2 + (y - k)^2 = r^2$. Expanded (general) form hides the center; recover it by completing the square in x and in y separately. Honors problems also ask you to find a circle through given points, decide whether a point lies inside/on/outside a circle (compare its distance to r), and find tangent lines at a point (the tangent is perpendicular to the radius there).

Practice

4.11 Find the center and radius of $x^2 + y^2 - 6x + 4y - 12 = 0$.

4.12 Write the equation of the circle with center $(-1, 4)$ passing through $(2, 8)$.

4.13 For the circle $(x - 3)^2 + (y + 2)^2 = 25$, find the slope of the tangent line at the point $(6, 2)$.

Arc length, sector area, and segment area (and radians)

For a central angle θ (in degrees) in a circle of radius r : arc length $= (\theta/360) \cdot 2\pi r$ and sector area $= (\theta/360) \cdot \pi r^2$. A circular SEGMENT (the region between a chord and its arc) has area $=$ sector area $-$ triangle area, where the triangle area is $\frac{1}{2}r^2 \sin \theta$. Many honors courses introduce radian measure here: θ in radians makes arc length simply $r\theta$ and sector area $\frac{1}{2}r^2\theta$, with $180^\circ = \pi$ radians.

Practice

4.14 A circle has radius 6. Find the arc length and sector area for a 60° central angle.

4.15 Find the area of the circular segment cut off by the chord of that same 60° arc ($r = 6$).

4.16 Convert 135° to radians, and find the arc length it subtends in a circle of radius 8.

Section 5: Trigonometry Extensions

Khan's geometry course covers right-triangle trig (SOH-CAH-TOA). Honors courses extend trig to ALL triangles via the Laws of Sines and Cosines, plus trig-based area formulas.

Law of Sines and the ambiguous (SSA) case

In any triangle, $a/\sin A = b/\sin B = c/\sin C$ (each ratio also equals $2R$, the circumdiameter). Use it when you know an angle and its opposite side (AAS, ASA, SSA). The SSA case is “ambiguous”: the given data may determine zero, one, or two triangles, because $\sin B = k$ has two solutions (B and $180^\circ - B$) between 0° and 180° . Always check whether the supplement also yields a valid triangle (angle sum under 180°).

Practice

- 5.1** In $\triangle ABC$, $m\angle A = 30^\circ$, $a = 6$, $b = 10$. Find all possible measures of $\angle B$.
- 5.2** In $\triangle ABC$, $m\angle A = 40^\circ$, $m\angle B = 60^\circ$, and $a = 12$. Find b to the nearest tenth.
- 5.3** In $\triangle ABC$, $m\angle A = 30^\circ$, $a = 4$, $b = 10$. How many triangles are possible?

Law of Cosines

$c^2 = a^2 + b^2 - 2ab \cdot \cos C$ generalizes the Pythagorean theorem ($\cos 90^\circ = 0$ collapses it to $c^2 = a^2 + b^2$). Use it for SAS (find the third side) and SSS (find any angle, by solving for the cosine). A useful check: if $c^2 > a^2 + b^2$ the angle at C is obtuse; if less, acute — the Law of Cosines is also the honors-level tool for classifying triangles by their sides.

Practice

- 5.4** Two sides of a triangle measure 5 and 7 with a 60° included angle. Find the third side.
- 5.5** A triangle has sides 7, 8, and 13. Find the largest angle to the nearest degree.
- 5.6** Classify the triangle with sides 6, 8, 9 as acute, right, or obtuse using the Law of Cosines idea.

Trig area formulas and Heron's formula

SAS area: $\text{Area} = \frac{1}{2}ab \cdot \sin C$, where C is the included angle — no height needed. SSS area (Heron): with semiperimeter $s = (a + b + c)/2$, $\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$. Honors courses combine these with the earlier tools: e.g., find a triangle's area from three sides, or the area of a regular polygon by slicing it into congruent SAS triangles at the center.

Practice

- 5.7** Find the area of a triangle with sides 8 and 5 and included angle 150° .
- 5.8** Use Heron's formula to find the area of a triangle with sides 5, 6, 7.
- 5.9** Find the area of a regular pentagon with circumradius 10 (distance from center to each vertex), using the SAS formula.

Inverse trig: solving for angles

Inverse trig functions ($\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, also written $\arcsin$ etc.) recover an angle from a ratio: if $\tan \theta = 3/4$ then $\theta = \tan^{-1}(3/4) \approx 36.87^\circ$. In right-triangle problems, pick the ratio that uses the two sides you know relative to the angle you want. Watch the calculator mode (degrees vs radians) — a classic source of wrong answers — and remember that in the ambiguous Law-of-Sines case, the calculator only returns the acute solution.

Practice

- 5.10** A right triangle has legs 3 and 4. Find the measure of the angle opposite the leg of length 3, to the nearest hundredth of a degree.
- 5.11** A 20-ft ladder leans against a wall, reaching 16 ft up. Find the angle the ladder makes with the ground.
- 5.12** A guy wire runs 50 ft from the top of a pole to a stake 30 ft from the pole's base. Find the wire's angle of elevation from the ground.

Section 6: Transformations as a Formal System

Modern honors courses (following the Common Core approach) **DEFINE** congruence and similarity through transformations, then reason about compositions. Khan covers individual transformations; the formal system is where the depth lives.

Compositions of transformations

Performing one transformation after another is a composition, and compositions of rigid motions have predictable results. Two reflections over **PARALLEL** lines produce a translation, perpendicular to the lines, through **TWICE** the distance between them. Two reflections over **INTERSECTING** lines produce a rotation about their intersection point through **TWICE** the angle between the lines. A glide reflection (reflection + translation along the mirror line) is the fourth and final type of rigid motion — every rigid motion of the plane is one of: translation, rotation, reflection, or glide reflection.

Practice

- 6.1** A figure is reflected over the line $y = 1$, then over the line $y = 4$. Describe the single transformation with the same effect.
- 6.2** A point is reflected over the x -axis, then over the y -axis. What single transformation results? Verify with the point $(3, 5)$.
- 6.3** Reflections over two lines that intersect at 35° produce what transformation?

Congruence and similarity via rigid motions and dilations

Two figures are **CONGRUENT** if and only if some sequence of rigid motions (translations, rotations, reflections) maps one exactly onto the other; they are **SIMILAR** if a sequence of rigid motions plus a dilation does. Honors tasks run in both directions: given two congruent figures, exhibit an explicit sequence of motions; or use the transformation definition to justify criteria like SAS. Coordinate rules to know cold: $(x, y) \rightarrow (x + a, y + b)$ translation; $(-x, y)$ reflection over y -axis; $(x, -y)$ over x -axis; (y, x) over $y = x$; $(-y, x)$ rotation 90° CCW; $(-x, -y)$ rotation 180° ; (kx, ky) dilation about the origin by factor k .

Practice

- 6.4** $\triangle ABC$ has vertices $A(1, 1)$, $B(4, 1)$, $C(1, 3)$. $\triangle DEF$ has vertices $D(-1, -1)$, $E(-4, -1)$, $F(-1, -3)$. Find a sequence of rigid motions mapping $\triangle ABC$ to $\triangle DEF$, and state what this proves.
- 6.5** $\triangle PQR$ has vertices $P(0, 0)$, $Q(2, 0)$, $R(0, 4)$. $\triangle P'Q'R'$ has vertices $P'(0, 0)$, $Q'(3, 0)$, $R'(0, 6)$. Are the triangles congruent, similar, or neither? Name the transformation.
- 6.6** What image does the rule $(x, y) \rightarrow (-y, x)$ produce for the point $(5, 2)$, and what transformation is it?

Symmetry of figures

A figure has line (reflection) symmetry if a reflection maps it onto itself, and rotational symmetry of order n if rotations of $360^\circ/n$ do. A regular n -gon has exactly n lines of symmetry and rotational symmetry of order n . Informally, the set of all symmetries of a figure forms its symmetry “group” — composing any two symmetries yields another. Honors questions ask you to count symmetries and find the smallest angle of rotational symmetry.

Practice

- 6.7** How many lines of symmetry does a regular pentagon have, and what is its smallest positive angle of rotational symmetry?

6.8 Which capital letters (in standard block form) have rotational symmetry of order 2 but NO line of symmetry?

6.9 A figure is carried onto itself by a 40° rotation about a point. What is the smallest possible order of its rotational symmetry?

Section 7: Three-Dimensional Geometry

Beyond plugging into volume formulas: honors courses ask WHERE the formulas come from (Cavalieri's principle), what cross-sections and solids of revolution look like, and how to handle composite and truncated solids.

Cavalieri's principle and the origin of volume formulas

Cavalieri's principle: if two solids have equal heights and equal cross-sectional areas at every level, they have equal volumes. Consequences: an oblique prism or cylinder has the same volume as a right one with the same base and height ($V = Bh$); every pyramid or cone has $V = \frac{1}{3}Bh$ regardless of where the apex sits; and the sphere formula $V = \frac{4}{3}\pi r^3$ can be derived by comparing a hemisphere to a cylinder with a cone removed. Expect at least one "explain using Cavalieri" question on an honors exam.

Practice

- 7.1** An oblique cylinder leans at 30° but has base radius 5 and (vertical) height 9. Find its volume and justify.
- 7.2** A stack of quarters is straight; a second stack uses the same 40 quarters but is skewed sideways. Compare the volumes and name the principle.
- 7.3** A pyramid and a prism have the same base area B and the same height h . What fraction of the prism's volume does the pyramid occupy?

Cross-sections and solids of revolution

A cross-section is the 2-D shape where a plane cuts a solid; the answer depends on the cutting angle (a plane can cut a cube into a triangle, rectangle, pentagon, or even a regular hexagon). A solid of revolution is generated by rotating a plane figure about a line: a rectangle sweeps a cylinder, a right triangle rotated about a leg sweeps a cone, a semicircle sweeps a sphere, and a figure rotated about an axis it doesn't touch sweeps a ring (annular) solid. Honors problems ask you both to identify the resulting solid and to compute its volume.

Practice

- 7.4** A right triangle with legs 3 and 4 is rotated about the leg of length 4. Identify the solid and find its volume.
- 7.5** What cross-sections are possible when a plane intersects a cylinder?
- 7.6** A 2×5 rectangle is rotated about one of its longer sides. Identify the resulting solid and find its volume.

Cones, frustums, spheres, and composite solids

Key formulas: cone $V = \frac{1}{3}\pi r^2 h$ with slant height $\ell = \sqrt{r^2 + h^2}$ and lateral area $\pi r \ell$; sphere $V = \frac{4}{3}\pi r^3$ and $SA = 4\pi r^2$; frustum (a cone with its top sliced off parallel to the base) $V = \frac{\pi h}{3}(R^2 + Rr + r^2)$. Composite solids (a cylinder capped by a hemisphere, a cone drilled out of a cylinder) are handled by adding and subtracting pieces — the skill is decomposing cleanly and not double-counting shared faces in surface-area problems.

Practice

- 7.7** A frustum has base radii 6 and 3 and height 4. Find its volume.

7.8 A grain silo is a cylinder of radius 3 and height 10 topped by a hemisphere. Find its total volume.

7.9 A sphere has volume 36π . Find its radius and surface area.

Euler's formula for polyhedra

For any convex polyhedron, $V - E + F = 2$ (vertices minus edges plus faces). It lets you deduce a missing count, and it underlies the classification of the five Platonic solids — the only convex polyhedra with congruent regular faces and identical vertices (tetrahedron, cube, octahedron, dodecahedron, icosahedron). Honors enrichment often includes verifying Euler's formula on each Platonic solid.

Practice

7.10 A convex polyhedron has 12 faces and 30 edges. How many vertices does it have, and which Platonic solid is it?

7.11 Verify Euler's formula for a square pyramid.

7.12 Each face of a polyhedron is a triangle, and it has 20 faces. Find E and V .

Section 8: Compass-and-Straightedge Constructions

Khan demonstrates several constructions, but honors classes require you to perform them AND prove why they work — usually via congruent triangles or the defining property of a perpendicular bisector or angle bisector.

Core constructions and their justifications

The essential toolkit: copy a segment/angle; bisect a segment (perpendicular bisector) and an angle; erect a perpendicular at a point on a line and drop one from an external point; construct a parallel through an external point (copy an angle to force congruent corresponding angles). The justifications almost always come down to SSS: the arcs guarantee equal radii, producing congruent triangles, which yield the equal angles or right angles you need. The deeper principle: the perpendicular bisector of AB is exactly the set of points equidistant from A and B , and an angle bisector is the set of points equidistant from the two sides.

Practice

- 8.1** Describe the construction of the perpendicular bisector of segment AB , and prove it works.
- 8.2** In the standard angle-bisector construction (arc from the vertex crossing both sides at X and Y , then equal arcs from X and Y meeting at Z), which congruence criterion proves ray VZ bisects $\angle V$?
- 8.3** How do you construct a line through point P parallel to line ℓ (P not on ℓ)?

Tangents, inscribed circles, and circumscribed circles

Tangent from an external point P to circle O : draw the midpoint M of OP , then the circle centered at M through O and P ; it meets circle O at the tangency points T . Why it works: $\angle OTP$ is inscribed in a semicircle of the auxiliary circle, hence a right angle, and a line perpendicular to a radius at its endpoint is tangent. Inscribed circle of a triangle: intersect two ANGLE bisectors to find the incenter, drop a perpendicular to any side for the radius. Circumscribed circle: intersect two PERPENDICULAR BISECTORS of the sides to find the circumcenter, whose distance to any vertex is the radius. Keeping “bisectors $\rightarrow$ incircle, perpendicular bisectors $\rightarrow$ circumcircle” straight is a favorite exam trap.

Practice

- 8.4** Explain why the semicircle (Thales) construction of a tangent from external point P actually produces a tangent line.
- 8.5** To circumscribe a circle about $\triangle ABC$, which lines do you construct, and why is their intersection equidistant from all three vertices?
- 8.6** Two tangent segments are drawn to a circle from the same external point. Prove they are congruent.

Regular polygons: the hexagon and beyond

A regular hexagon is constructible because its side length equals its circumradius: draw a circle, keep the same compass opening, and step around the circle six times — the marks are the vertices. Connecting every other mark gives an equilateral triangle; perpendicular diameters give a square; bisecting central angles doubles any construction (hexagon $\rightarrow$ 12-gon). Enrichment fact honors teachers love: Gauss proved which regular polygons are constructible — the 17-gon is, the 7-gon and 9-gon are not.

Practice

- 8.7** Why does stepping the compass (set to the radius) around a circle exactly six times construct a regular hexagon?
- 8.8** Given a circle, how do you construct an inscribed square?
- 8.9** A regular hexagon is inscribed in a circle of radius 8. Find its perimeter and its area.

Section 9: Coordinate Geometry Synthesis

Honors courses use the coordinate plane not just for computation but for proving general theorems — which requires clever placement of figures and a few formulas Khan treats briefly.

Partitioning a segment in a given ratio

To find the point P that partitions segment AB in ratio $m:n$ from A , travel the fraction $m/(m+n)$ of the way: $P = (x_1 + (m/(m+n))(x_2 - x_1), y_1 + (m/(m+n))(y_2 - y_1))$. The midpoint is the special case 1:1. Watch the direction — partitioning “from A to B ” in ratio 2:1 is a different point than “from B to A ” in ratio 2:1, and mixing these up is the most common error.

Practice

- 9.1** Find the point that partitions the segment from $A(1, 2)$ to $B(7, 11)$ in the ratio 2:1.
- 9.2** Find the point one-quarter of the way from $M(-4, 6)$ to $N(8, -2)$.
- 9.3** Point $P(3, 4)$ lies on segment AB with $A(0, 1)$. If $AP:PB = 3:2$, find B .

Distance from a point to a line

The distance from a point to a line is measured along the perpendicular. Two methods: (1) find the perpendicular through the point, solve for the intersection (foot), and apply the distance formula; (2) use the direct formula — for line $ax + by + c = 0$ and point (x_0, y_0) , $d = |ax_0 + by_0 + c| / \sqrt{a^2 + b^2}$. This also gives the distance between parallel lines (pick any point on one) and shows up inside problems about tangent circles and areas of triangles with a known base line.

Practice

- 9.4** Find the distance from the point $(3, 4)$ to the line $3x + 4y - 10 = 0$.
- 9.5** Find the distance between the parallel lines $y = 2x + 1$ and $y = 2x + 6$.
- 9.6** A circle is centered at $(2, -1)$. Find its radius if it is tangent to the line $4x - 3y + 14 = 0$.

Proving general theorems with strategic placement

To prove a theorem for ALL triangles or quadrilaterals, place a GENERAL figure using variables — and place it cleverly: one vertex at the origin, one side on the x -axis, and coordinates like $(2a, 0)$ and $(2b, 2c)$ so that midpoints come out fraction-free. The proof must never use specific numbers. Standard honors targets: the midsegment theorem, “the diagonals of a rectangle are congruent,” “the medians of a triangle are concurrent,” and “the midpoints of any quadrilateral’s sides form a parallelogram” (Varignon’s theorem).

Practice

- 9.7** Set up coordinates and prove: the diagonals of a rectangle are congruent.
- 9.8** Prove the median to the hypotenuse of a right triangle is half the hypotenuse, using coordinates.
- 9.9** Quadrilateral $ABCD$ has vertices $A(0,0)$, $B(2a, 0)$, $C(2b, 2c)$, $D(2d, 2e)$ — completely general. Show the midpoints of its sides form a parallelogram (Varignon’s theorem).

Section 10: Honors-Level Extras and Enrichment

Content varies by school, but these enrichment topics appear often enough on honors finals — and in math competitions — to be worth a pass.

A taste of non-Euclidean geometry

Euclidean geometry rests on the Parallel Postulate: through a point not on a line there is exactly ONE parallel. Deny it and consistent geometries still emerge. Spherical (elliptic) geometry: “lines” are great circles, there are NO parallels, and triangle angle sums EXCEED 180° (the excess is proportional to area). Hyperbolic geometry: infinitely many parallels through the point, and angle sums are LESS than 180° . The takeaway honors courses want: theorems depend on axioms, and familiar facts like “angle sum = 180° ” are consequences of the parallel postulate, not universal truths.

Practice

- 10.1** On a globe, a traveler goes from the North Pole to the equator, a quarter of the way around the equator, then back to the pole. What are the angles of this spherical triangle, and what is their sum?
- 10.2** In which geometry — Euclidean, spherical, or hyperbolic — can a triangle have angle sum 150° ? Angle sum exactly 180° ? No parallel lines at all?
- 10.3** Which single postulate distinguishes these three geometries?

Locus problems

A locus is the set of ALL points satisfying a condition. Core catalog: equidistant from two points → the perpendicular bisector of their segment; equidistant from two intersecting lines → the pair of angle bisectors; equidistant from two parallel lines → the parallel midline; at fixed distance r from a point → a circle; at fixed distance from a line → two parallel lines; equidistant from a point and a line → a parabola (the honors preview of conics). Compound locus problems intersect two conditions — sketch each locus, then count intersection points.

Practice

- 10.4** Describe the locus of points equidistant from A and B and also 3 units from the midpoint of AB, if $AB = 10$.
- 10.5** What is the locus of points equidistant from two parallel lines 8 units apart, and its distance from each line?
- 10.6** How many points are equidistant from two intersecting lines AND 5 units from their intersection point?

Geometric probability

When outcomes correspond to points in a region, probability = (measure of favorable region) / (measure of total region), using length for segments, area for plane regions. The method converts probability questions into the area formulas from earlier chapters — circles inside squares, sectors of spinners, “waiting time” segments on a number line.

Practice

- 10.7** A dart lands at a random point inside a 4×4 square. A circle of radius 1 is centered at the square’s center. Find the probability the dart lands inside the circle.

10.8 A point is chosen at random on a segment of length 12. Find the probability it lies within 2 units of the midpoint.

10.9 A dartboard is a circle of radius 9 with a bullseye of radius 3. Find the probability a random hit on the board lands in the bullseye.

The golden ratio, Ptolemy's theorem, and classic enrichment

The golden ratio $\phi = (1 + \sqrt{5})/2 \approx 1.618$ satisfies $\phi^2 = \phi + 1$; it is the ratio of a regular pentagon's diagonal to its side, and a "golden rectangle" keeps its proportions when a square is removed.

Ptolemy's theorem: in a cyclic quadrilateral, (product of the diagonals) = (sum of the products of opposite sides): $AC \cdot BD = AB \cdot CD + AD \cdot BC$ — applying it to a rectangle recovers the Pythagorean theorem. These, along with locus and geometric probability, are the classic "challenge problem" well from which honors finals draw.

Practice

10.10 Show that $\phi = (1 + \sqrt{5})/2$ satisfies the defining equation $\phi^2 = \phi + 1$.

10.11 Verify Ptolemy's theorem on a square of side 1.

10.12 Apply Ptolemy's theorem to a rectangle with sides a and b and diagonal d to derive a famous theorem.

Answer Key

Answers give the key steps, not always every written line — for proofs, any logically complete argument with correct justifications earns full credit.

Section 1: Formal Proof Writing

1.1 Key chain: $\angle 1$ and $\angle 3$ form a linear pair, so $m\angle 1 + m\angle 3 = 180^\circ$ (Linear Pair Postulate). Likewise $m\angle 2 + m\angle 3 = 180^\circ$. Therefore $m\angle 1 + m\angle 3 = m\angle 2 + m\angle 3$ (substitution), so $m\angle 1 = m\angle 2$ (subtraction property), hence $\angle 1 \cong \angle 2$.

1.2 Since $\angle A$ is supplementary to $\angle C$, $m\angle A + m\angle C = 180^\circ$; since $\angle B$ is supplementary to $\angle C$, $m\angle B + m\angle C = 180^\circ$. By substitution $m\angle A + m\angle C = m\angle B + m\angle C$, and subtracting $m\angle C$ from both sides gives $m\angle A = m\angle B$, so $\angle A \cong \angle B$. (This is the Congruent Supplements Theorem.)

1.3 $AD = \frac{1}{2}AB$ and $AE = \frac{1}{2}AC$ (definition of midpoint). Since $AB = AC$, halves of equal quantities are equal, so $AD = AE$. A triangle with two congruent sides is isosceles by definition.

1.4 Assume a triangle has two right angles. Then those two angles alone sum to $90^\circ + 90^\circ = 180^\circ$, forcing the third angle to measure 0° — contradicting the fact that the three angle measures of a triangle are positive and sum to exactly 180° . So the assumption is false: at most one right angle.

1.5 Assume the lines are parallel. Then by the Alternate Interior Angles Theorem the alternate interior angles must be congruent — contradicting the given that they are not congruent. Hence the lines are not parallel.

1.6 “Assume $\angle B \cong \angle C$.” (You then derive $AB = AC$ by the Converse of the Base Angles Theorem, contradicting $AB \neq AC$.)

1.7 $AB = \sqrt{4^2 + 2^2} = \sqrt{20}$; $BC = \sqrt{(-2)^2 + 4^2} = \sqrt{20}$; $CD = \sqrt{(-4)^2 + (-2)^2} = \sqrt{20}$; $DA = \sqrt{2^2 + (-4)^2} = \sqrt{20}$ — all sides congruent, so ABCD is a rhombus. Slope $AB = \frac{2}{4} = \frac{1}{2}$ and slope $BC = \frac{4}{-2} = -2$; their product is -1 , so $\angle B$ is a right angle. A rhombus with a right angle is a square.

1.8 $PR = \sqrt{3^2 + 5^2} = \sqrt{34}$ and $QR = \sqrt{(6-3)^2 + 5^2} = \sqrt{34}$. Two sides are congruent, so the triangle is isosceles. ($PQ = 6 \neq \sqrt{34}$, so it is not equilateral.)

1.9 Show both diagonals have the same midpoint. Place the parallelogram at $A(0,0)$, $B(a,0)$, $C(a+b,c)$, $D(b,c)$; the midpoint of AC is $((a+b)/2, c/2)$ and the midpoint of BD is $((a+b)/2, c/2)$ — identical, so the diagonals bisect each other.

1.10 Converse: “If a quadrilateral has four right angles, then it is a square” — false (a non-square rectangle). Inverse: “If a quadrilateral is not a square, then it does not have four right angles” — false (same counterexample). Contrapositive: “If a quadrilateral does not have four right angles, then it is not a square” — true (equivalent to the original).

1.11 No. “Vertical $\Rightarrow$ congruent” is true, but the converse “congruent $\Rightarrow$ vertical” is false (two separate 40° angles are congruent without being vertical). A biconditional requires both directions to hold.

1.12 (a) undefined term; (b) postulate; (c) definition; (d) theorem (it must be proved, e.g., using SAS).

Section 2: Triangle Centers and Advanced Triangle Relationships

2.1 Hospital: circumcenter (equidistant from vertices). Fountain: incenter (equidistant from sides).

2.2 $AG = (2/3)(18) = 12$ and $GM = (1/3)(18) = 6$, since the centroid divides each median 2:1 from the vertex.

2.3 Circumcenter: the midpoint of the hypotenuse (so the circumradius is half the hypotenuse — a fact worth memorizing). Orthocenter: at the right-angle vertex itself, since the two legs are altitudes.

2.4 G divides OH with $OG:GH = 1:2$, so $G = O + (1/3)(H - O) = (2, 1)$.

2.5 The incenter. It lies on the Euler line only for isosceles triangles (for equilateral triangles all centers coincide at one point).

2.6 $5 < x < 21$ (from $|13 - 8| < x < 13 + 8$).

2.7 $DF > AC$ by the Hinge Theorem — with two pairs of sides equal, the larger included angle (70°) opens onto the longer opposite side.

2.8 $\angle Y$ (opposite $XZ = 7$) $< \angle Z$ (opposite $XY = 9$) $< \angle X$ (opposite $YZ = 12$).

2.9 $DE = 7$ (half of BC). Since $DE \parallel BC$, $\angle ADE \cong \angle ABC$ (corresponding angles), so $m\angle ADE = 63^\circ$.

2.10 Perimeter of $\triangle PQR = 44$ (each midsegment is half a side). The medial triangle has $\frac{1}{4}$ of the area.

2.11 Midpoints: $D =$ midpoint of $AB = (b, 0)$ and $E =$ midpoint of $AC = (c, d)$. Slope $DE = (d - 0)/(c - b)$ and slope $BC = (2d - 0)/(2c - 2b) = d/(c - b)$ — equal slopes, so $DE \parallel BC$. Lengths: $DE = \sqrt{(c-b)^2 + d^2}$ while $BC = \sqrt{(2c-2b)^2 + (2d)^2} = 2\sqrt{(c-b)^2 + d^2} = 2 \cdot DE$.

Section 3: Similarity Beyond the Basics

3.1 $h = \sqrt{4 \cdot 9} = 6$.

3.2 $\text{leg} = \sqrt{c \cdot m} = \sqrt{13 \cdot 4} = 2\sqrt{13}$.

3.3 $\triangle ABC \sim \triangle ACD \sim \triangle CBD$ (each shares an acute angle with the original plus a right angle, so AA applies).

3.4 $AD/DB = AE/EC \rightarrow 6/4 = 9/EC \rightarrow EC = 6$.

3.5 $BD/DC = AB/AC = 8/12 = 2/3$, and $BD + DC = 15$, so $BD = 6$ and $DC = 9$.

3.6 $5/8 = 7.5/x \rightarrow x = 12$.

3.7 $\angle A \cong \angle E$ and $\angle B \cong \angle D$ (alternate interior angles, $AB \parallel DE$), so $\triangle ABC \sim \triangle EDC$ by AA. Then $AB/ED = AC/EC \rightarrow 10/ED = 8/12 \rightarrow DE = 15$.

3.8 Area ratio $= (3/5)^2 = 9/25$, so larger area $= 27 \cdot 25/9 = 75$.

3.9 $AP/AB = 4/10 = 2/5$ and $AQ/AC = 6/15 = 2/5$. With the shared $\angle A$, $\triangle APQ \sim \triangle ABC$ by SAS~, so $\angle APQ \cong \angle ABC$ and $PQ \parallel BC$ (converse of Side-Splitter also works).

Section 4: Circles — The Deep End

4.1 Central $\angle AOB = 100^\circ$; inscribed $\angle ACB = 50^\circ$.

4.2 Opposite angles are supplementary: $(x + 10) + (2x - 40) = 180 \rightarrow 3x = 210 \rightarrow x = 70$. So $m\angle A = 80^\circ$ and $m\angle C = 100^\circ$.

4.3 $\angle ACB = 90^\circ$ (angle in a semicircle), so $m\angle ABC = 180^\circ - 90^\circ - 37^\circ = 53^\circ$.

4.4 $\frac{1}{2}(80 + 40) = 60^\circ$.

4.5 $\frac{1}{2}(120 - 40) = 40^\circ$.

4.6 $\frac{1}{2}(150) = 75^\circ$.

4.7 The arcs sum to 360° and their difference satisfies $\frac{1}{2}(\text{far} - \text{near}) = 40^\circ$, so $\text{far} - \text{near} = 80^\circ$. Solving: $\text{far} = 220^\circ$, $\text{near} = 140^\circ$.

4.8 $3 \cdot 8 = 4 \cdot PD \rightarrow PD = 6$.

4.9 $t^2 = 4 \cdot 9 = 36 \rightarrow t = 6$.

4.10 $5(5 + 7) = 6(6 + x) \rightarrow 60 = 36 + 6x \rightarrow x = 4$.

4.11 $(x^2 - 6x + 9) + (y^2 + 4y + 4) = 12 + 9 + 4 \rightarrow (x - 3)^2 + (y + 2)^2 = 25$. Center $(3, -2)$, radius 5.

4.12 $r^2 = (2 - (-1))^2 + (8 - 4)^2 = 9 + 16 = 25$, so $(x + 1)^2 + (y - 4)^2 = 25$.

4.13 Radius slope from $(3, -2)$ to $(6, 2)$ is $4/3$; the tangent is perpendicular, slope $-3/4$.

4.14 Arc $= (60/360)(2\pi \cdot 6) = 2\pi$. Sector $= (60/360)(\pi \cdot 36) = 6\pi$.

4.15 Segment = sector – triangle = $6\pi - \frac{1}{2}(6^2)\sin 60^\circ = 6\pi - 9\sqrt{3} \approx 3.26$.

4.16 $135^\circ = 3\pi/4$ radians; arc length = $r\theta = 8 \cdot 3\pi/4 = 6\pi$.

Section 5: Trigonometry Extensions

5.1 $\sin B = b \cdot \sin A / a = 10(0.5)/6 = 5/6$, so $B \approx 56.4^\circ$ or $B \approx 123.6^\circ$. Both work ($30^\circ + 56.4^\circ < 180^\circ$ and $30^\circ + 123.6^\circ < 180^\circ$), so TWO triangles exist — the classic ambiguous case.

5.2 $b = a \cdot \sin B / \sin A = 12 \cdot \sin 60^\circ / \sin 40^\circ \approx 12(0.8660)/0.6428 \approx 16.2$.

5.3 $\sin B = 10(0.5)/4 = 1.25 > 1$ — impossible. Zero triangles.

5.4 $c^2 = 25 + 49 - 2(5)(7)\cos 60^\circ = 74 - 35 = 39$, so $c = \sqrt{39} \approx 6.24$.

5.5 Largest angle is opposite 13: $\cos C = (7^2 + 8^2 - 13^2)/(2 \cdot 7 \cdot 8) = (49 + 64 - 169)/112 = -0.5$, so $C = 120^\circ$.

5.6 $9^2 = 81$ vs $6^2 + 8^2 = 100$. Since $81 < 100$, the largest angle is acute — the triangle is acute.

5.7 Area = $\frac{1}{2}(8)(5)\sin 150^\circ = 20(0.5) = 10$.

5.8 $s = 9$; Area = $\sqrt{(9 \cdot 4 \cdot 3 \cdot 2)} = \sqrt{216} = 6\sqrt{6} \approx 14.7$.

5.9 Five central triangles, each with two sides of 10 and included angle 72° : Area = $5 \cdot \frac{1}{2}(10)(10)\sin 72^\circ \approx 250(0.9511) \approx 237.8$.

5.10 $\theta = \tan^{-1}(3/4) \approx 36.87^\circ$.

5.11 $\sin \theta = 16/20 = 0.8 \rightarrow \theta = \sin^{-1}(0.8) \approx 53.13^\circ$.

5.12 $\cos \theta = 30/50 = 0.6 \rightarrow \theta = \cos^{-1}(0.6) \approx 53.13^\circ$.

Section 6: Transformations as a Formal System

6.1 A translation 6 units up (twice the distance 3 between the parallel lines, in the direction from the first line toward the second).

6.2 A 180° rotation about the origin (the lines meet at 90° , and $2 \cdot 90^\circ = 180^\circ$). Check: $(3, 5) \rightarrow (3, -5) \rightarrow (-3, -5) = 180^\circ$ rotation of $(3, 5)$.

6.3 A rotation of 70° about the intersection point.

6.4 A single 180° rotation about the origin, $(x, y) \rightarrow (-x, -y)$, maps $A \rightarrow D$, $B \rightarrow E$, $C \rightarrow F$. Since a rigid motion maps one triangle onto the other, $\triangle ABC \cong \triangle DEF$.

6.5 Similar but not congruent: the dilation $(x, y) \rightarrow (1.5x, 1.5y)$ centered at the origin maps one to the other (scale factor $3/2$).

6.6 $(-2, 5)$; a 90° counterclockwise rotation about the origin.

6.7 5 lines; smallest rotation $360^\circ/5 = 72^\circ$.

6.8 N, S, and Z — each maps to itself under a 180° rotation but has no mirror line.

6.9 Rotations by 40° generate rotations by $40^\circ, 80^\circ, \dots, 360^\circ$, giving order 9 (since $360/40 = 9$).

Section 7: Three-Dimensional Geometry

7.1 $V = \pi r^2 h = 225\pi$. By Cavalieri, every horizontal cross-section is a circle of area 25π — the same as an upright cylinder's — so the slant doesn't change the volume.

7.2 Equal volumes — the cross-sections at each height are identical coins. This is Cavalieri's principle in miniature.

7.3 One-third: $V(\text{pyramid}) = \frac{1}{3}Bh$ vs $V(\text{prism}) = Bh$.

7.4 A cone with radius 3 and height 4: $V = \frac{1}{3}\pi(3^2)(4) = 12\pi$.

7.5 A circle (parallel to the base), an ellipse (tilted), a rectangle (parallel to the axis), and degenerate cases (a line or point when tangent).

7.6 A cylinder of radius 2 and height 5: $V = \pi(2^2)(5) = 20\pi$.

7.7 $V = (\pi \cdot 4/3)(36 + 18 + 9) = (4\pi/3)(63) = 84\pi$.

7.8 $V = \pi(9)(10) + \frac{1}{2} \cdot (4/3)\pi(27) = 90\pi + 18\pi = 108\pi$.

7.9 $(4/3)\pi r^3 = 36\pi \rightarrow r^3 = 27 \rightarrow r = 3$; $SA = 4\pi(9) = 36\pi$.

7.10 $V = 2 + E - F = 2 + 30 - 12 = 20$ — the regular dodecahedron.

7.11 $V = 5$, $E = 8$, $F = 5$: $5 - 8 + 5 = 2$. ✓

7.12 $E = (20 \cdot 3)/2 = 30$ (each edge shared by two faces); $V = 2 + 30 - 20 = 12$ — the icosahedron.

Section 8: Compass-and-Straightedge Constructions

8.1 With the same radius $r > \frac{1}{2}AB$, draw arcs centered at A and B meeting at P and Q; draw line PQ. Proof: $PA = PB = QA = QB = r$, so P and Q are each equidistant from A and B; the set of such points is the perpendicular bisector, so line PQ is it. (Equivalently, APBQ is a rhombus and its diagonals are perpendicular bisectors of each other.)

8.2 SSS: $VX = VY$ (same first arc), $XZ = YZ$ (equal second arcs), $VZ = VZ$. So $\triangle VXZ \cong \triangle VYZ$, giving $\angle XVZ \cong \angle YVZ$.

8.3 Draw any transversal through P crossing ℓ , then copy the angle the transversal makes with ℓ to the corresponding position at P. Congruent corresponding angles force the new line parallel to ℓ (Corresponding Angles Converse).

8.4 T lies on the circle with diameter OP, so $\angle OTP$ is inscribed in a semicircle and measures 90° . Thus $PT \perp OT$ at T, a point of circle O — and a line perpendicular to a radius at its outer endpoint is tangent to the circle.

8.5 Perpendicular bisectors of two sides (say AB and BC). Any point on the perpendicular bisector of AB satisfies $PA = PB$, and on that of BC satisfies $PB = PC$; the intersection satisfies $PA = PB = PC$, so a circle centered there through A passes through all three.

8.6 Join the center O to the external point P and to the tangency points S, T. $\triangle OSP \cong \triangle OTP$ by HL (OS = OT are radii, OP shared, right angles at S and T), so $PS \cong PT$.

8.7 Each chord equals the radius, so each central triangle has three sides equal to r — equilateral, with a 60° central angle. Six 60° angles complete 360° exactly, spacing six vertices evenly.

8.8 Draw a diameter, construct its perpendicular bisector (a second diameter $\perp$ the first), and connect the four points where the diameters meet the circle.

8.9 Side = radius = 8, perimeter 48. Area = 6 equilateral triangles = $6 \cdot (\sqrt{3}/4)(64) = 96\sqrt{3} \approx 166.3$.

Section 9: Coordinate Geometry Synthesis

9.1 Go $2/3$ of the way: $P = (1 + \frac{2}{3} \cdot 6, 2 + \frac{2}{3} \cdot 9) = (5, 8)$.

9.2 $P = (-4 + \frac{1}{4} \cdot 12, 6 + \frac{1}{4} \cdot (-8)) = (-1, 4)$.

9.3 P is $3/5$ of the way: $(3, 4) = (0 + (3/5)(x - 0), 1 + (3/5)(y - 1)) \rightarrow x = 5, y = 6$. B(5, 6).

9.4 $d = |3 \cdot 3 + 4 \cdot 4 - 10| / \sqrt{(9 + 16)} = |15| / 5 = 3$.

9.5 Rewrite: $2x - y + 1 = 0$; take (0, 6) on the second line: $d = |2 \cdot 0 - 6 + 1| / \sqrt{5} = 5 / \sqrt{5} = \sqrt{5} \approx 2.24$.

9.6 r = distance from center to line = $|4 \cdot 2 - 3(-1) + 14| / \sqrt{(16 + 9)} = |25| / 5 = 5$.

9.7 Place the rectangle at A(0, 0), B(a, 0), C(a, b), D(0, b). $AC = \sqrt{(a^2 + b^2)}$ and $BD = \sqrt{((0 - a)^2 + (b - 0)^2)} = \sqrt{(a^2 + b^2)}$. Equal, so the diagonals are congruent.

9.8 Place the right angle at C(0, 0), with A(2a, 0), B(0, 2b). The hypotenuse midpoint is M(a, b). $CM = \sqrt{(a^2 + b^2)}$, while $AB = \sqrt{(4a^2 + 4b^2)} = 2\sqrt{(a^2 + b^2)}$. So $CM = \frac{1}{2}AB$.

9.9 Midpoints: P(a, 0), Q(a + b, c), R(b + d, c + e), S(d, e). Slope PQ = c/b and slope SR = $(c + e - e) / ((b + d) - d) = c/b$ — parallel. Slope QR = $((c + e) - c) / ((b + d) - (a + b)) = e / (d - a)$ and slope PS = $(e - 0) / (d - a) = e / (d - a)$ — parallel. Both pairs of opposite sides parallel $\rightarrow$ parallelogram.

Section 10: Honors-Level Extras and Enrichment

10.1 All three angles are 90° (each turn is a right angle), summing to 270° — impossible in the Euclidean plane, routine on a sphere.

10.2 150° : hyperbolic. Exactly 180° : Euclidean. No parallels: spherical.

10.3 The Parallel Postulate: exactly one parallel (Euclidean), none (spherical), infinitely many (hyperbolic).

10.4 The first condition gives the perpendicular bisector of AB; the second gives a circle of radius 3 centered at the midpoint. They intersect in exactly 2 points (one on each side of AB).

10.5 The line parallel to both, midway between them — 4 units from each.

10.6 Four. The equidistant locus is the two angle-bisector lines through the intersection; the circle of radius 5 meets them in 4 points.

10.7 $\pi(1^2) / 16 = \pi/16 \approx 0.196$.

10.8 Favorable length = 4 (from 4 to 8); probability $4/12 = 1/3$.

10.9 $9\pi / 81\pi = 1/9$.

10.10 $\phi^2 = (1 + \sqrt{5})^2/4 = (6 + 2\sqrt{5})/4 = (3 + \sqrt{5})/2$, and $\phi + 1 = (1 + \sqrt{5})/2 + 2/2 = (3 + \sqrt{5})/2$. Equal. ✓

10.11 Diagonals both $\sqrt{2}$: product = 2. Opposite sides: $1 \cdot 1 + 1 \cdot 1 = 2$. Equal. ✓

10.12 A rectangle is cyclic with both diagonals equal to d: $d \cdot d = a \cdot a + b \cdot b$, i.e., $d^2 = a^2 + b^2$ — the Pythagorean theorem.