

Sequences & Series: The Big Words

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A list with a secret rule - and what happens when you add it up

SEQUENCE

A **sequence** is a list of numbers in a specific order that follows a **secret rule**. If the numbers are random (7, 2, 99, 4) it's just a list. If there's a rule, it's a sequence - and your job is to crack the code!



Rule: **add 3**. Each number is called a **term**. The 1st term is 2, the 2nd term is 5, the 5th term is 14. We write the **nth** term to mean 'whichever term you're asking about'.

SERIES

A **series** is what you get when you **add up** the terms of a sequence. Same numbers, but now with plus signs between them - and usually we want to know the total.

$$2 + 5 + 8 + 11 + 14 = 40$$

The series has 5 terms. Its SUM is 40.

Sequence = the list. **Series** = the list added up. Contest problems ask things like 'What is the 20th term?' (sequence) or 'What is the sum of the first 20 terms?' (series).

Words you'll see in problems

term - one number in the list **first term** - the starting number **common difference** - the gap you add each time **common ratio** - the number you multiply by each time **sum** - everything added together

TIP A sequence is like a staircase; a series is how high you've climbed after taking all the steps.

Your Turn: Sequence or Series?

Show you know the words

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Name: _____

Part A: Circle SEQUENCE or SERIES

2, 4, 6, 8, 10 SEQUENCE / SERIES

$2 + 4 + 6 + 8 + 10$ SEQUENCE / SERIES

100, 90, 80, 70 SEQUENCE / SERIES

$1 + 3 + 9 + 27$ SEQUENCE / SERIES

5, 10, 20, 40, 80 SEQUENCE / SERIES

$7 + 7 + 7 + 7$ SEQUENCE / SERIES

Bonus: what is the sum of the series in the last one? _____

Part B: Sequence detective

For the sequence 4, 7, 10, 13, 16, 19, ...

1. What is the 1st term? _____
2. What is the 4th term? _____
3. What is the common difference (the gap)? _____
4. What is the 7th term? _____
5. Write the series for the first 4 terms: _____ + _____ + _____ + _____ = _____

Part C: Make your own

Write a sequence with the rule 'add 5' starting at 2: _____, _____, _____, _____, _____

Now write it as a series and find the sum: _____ + _____ + _____ + _____ + _____ = _____

TIP Commas = sequence. Plus signs = series. If you see the word 'sum' or 'total', it's a series question.

The Bridge Strategy

How to crack the code: draw bridges and write the jump

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The strategy

- 1 Draw a little **bridge** (an arc) between each pair of neighbors and write the jump on it: +2, -1, x3...
- 2 Look at the row of bridges. All the same? You've cracked it! Use the rule to find the next number.
- 3 Not the same? Look at the jumps themselves. Do THEY make a pattern (+1, +2, +3... or x2, x2...)? Then continue the pattern of jumps.

One row of bridges is enough



Bridges in a pattern



The bridges aren't all the same - but they go +1, +2, +3, +4. That's a pattern! So the next jump is +5, and the next number is $12 + 5 = 17$...

Try multiplying when adding doesn't work



Gaps of +3, +6, +12, +24 are doubling - a sign to try **x2** instead. Rule: multiply by 2. Next: 96.

TIP Always draw the bridges. Even easy sequences can trick you - the bridges make the rule impossible to miss.

Your Turn: Bridges

Draw the bridges, write the jumps, find the next two numbers

Name: _____

1

Rule: _____

7

11

15

19

23

2

Rule: _____

50

44

38

32

26

3

Rule: _____

1

3

7

13

21

4

Rule: _____

2

6

18

54

5

Rule: _____

100

50

25

6

Rule: _____

1

4

9

16

25

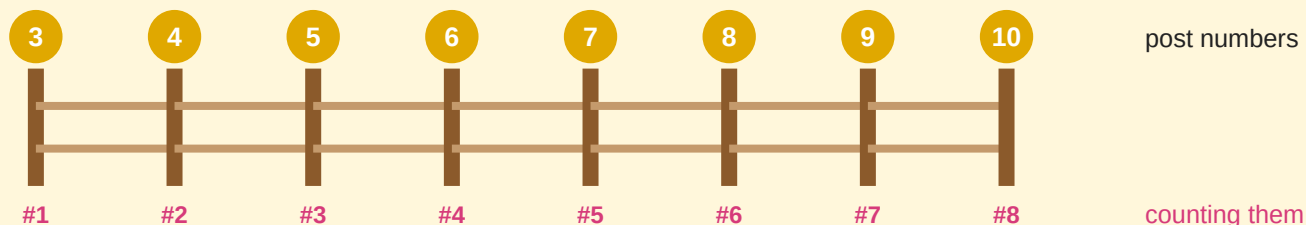
The Fencepost Problem

How many posts from post a to post b? (It's NOT $b - a$!)

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The trap

A fence runs from post 3 to post 10. How many posts is that? Most people say $10 - 3 = 7$. **Count them:**



8 posts, not 7! The gaps (sections of fence) are 7, but the posts are one more than the gaps.

The formula

$b - a$ counts the **jumps** from a to b . But the first post was already standing before you jumped at all - so add it back:

$$\text{posts from } a \text{ to } b = b - a + 1$$

Posts 3 to 10: $10 - 3 + 1 = 8$. Pages 12 to 40: $40 - 12 + 1 = 29$ pages. Numbers 1 to 100: $100 - 1 + 1 = 100$. Gaps (sections of fence) = $b - a$, posts = $b - a + 1$.

Why it matters for sequences

Counting terms: How many terms are in 5, 10, 15, ..., 50? Those are 5×1 up to 5×10 , so positions 1 to 10: $10 - 1 + 1 = 10$ terms (not 9!). You need n to use the sum formula.

The n th-term formula: from the 1st term to the n th there are $n - 1$ jumps, which is why the formula says $d \times (n - 1)$. Jumps = posts - 1, every time.

Your turn

1. How many whole numbers from 17 to 65? _____
2. You read pages 48 through 91 of a book. How many pages did you read? _____
3. How many multiples of 3 from 3 to 60? _____ (Hint: they're 3×1 up to 3×20 .)
4. A fence has 12 posts. How many sections of fence between them? _____

TIP Whenever you count 'from here to there', ask: am I counting the posts or the gaps? Posts = gaps + 1.

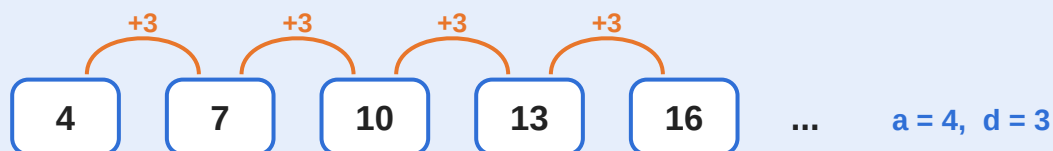
The Steady Jumper (Arithmetic)

Same jump every time - and a formula so you never count 100 steps

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What it is

An **arithmetic sequence** adds (or subtracts) the **same amount** every time. That amount is the **common difference, d**. The first term is called **a**.



Finding any term: the nth-term formula

To get from the 1st term to the **n**th term you take **n - 1** jumps of size **d**:

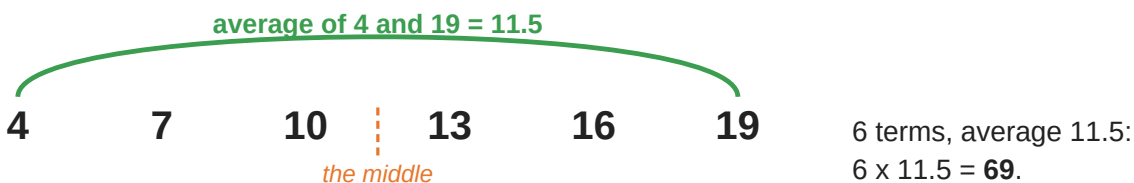
$$\text{nth term} = a + d \times (n - 1)$$

Example: What is the 50th term of 4, 7, 10, 13, ...? $a = 4, d = 3, n = 50$. $4 + 3 \times (50 - 1) = 4 + 3 \times 49 = 4 + 147 = 151$.

Why $n - 1$? From term 1 to term 50 is 49 jumps, not 50 - count the fence posts, not the fences!

Adding it all up: the arithmetic series

In a steady jumper, the terms are evenly spread out - so the **average** of all of them is just the **average of the first and last**. And any sum = average $\times$ how many.



$$\text{Sum} = (\text{first} + \text{last}) / 2 \times n$$

Example: $4 + 7 + 10 + \dots + 19$ has $n = 6$ terms. Average = $(4 + 19) / 2 = 11.5$. $11.5 \times 6 = 69$. If you only know a, d and n , find the last term with the nth-term formula first.

TIP Famous one: $1 + 2 + 3 + \dots + 100$: average is $(1 + 100) / 2 = 50.5$, times 100 terms = 5050. A boy named Gauss did this in seconds when his teacher wanted to keep the class busy!

Your Turn: Steady Jumpers

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Use the formulas - write a , d , and n first

Name: _____

$\text{nth term} = a + d \times (n - 1)$

$\text{Sum} = (\text{first} + \text{last}) / 2 \times n$

How many terms = last position - first position + 1

1 Find the 25th term of 6, 10, 14, 18, ...

$a = \underline{\hspace{1cm}}$ $d = \underline{\hspace{1cm}}$ $n = \underline{\hspace{1cm}}$

Answer: _____

2 Find the 40th term of 100, 97, 94, 91, ...

$a = \underline{\hspace{1cm}}$ $d = \underline{\hspace{1cm}}$ $n = \underline{\hspace{1cm}}$

Answer: _____

3 Find the sum: $5 + 10 + 15 + \dots + 50$

How many terms? Count them: $5 \times 1, 5 \times 2, \dots 5 \times 10$.

TIP Check: does your nth term make sense? The 25th term of a sequence that grows by 4 should be about 100 more than the first term.

Your Turn: Steady Jumpers

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Use the formulas - write a , d , and n first

Name: _____

$\text{nth term} = a + d \times (n - 1)$

$\text{Sum} = (\text{first} + \text{last}) / 2 \times n$

How many terms = last position - first position + 1

- 4 Find the sum of the first 20 terms of 3, 7, 11, 15, ...

Find the 20th term first, then use the sum formula.

Answer: _____

- 5 A dragon hoards 12 coins on day 1, 15 on day 2, 18 on day 3... How many coins does it ADD on day 30? How many does it have in total after 30 days?

Two questions: a term, then a sum.

Answer: _____

- 6 Find the sum $1 + 2 + 3 + \dots + 60$

Be like Gauss!

TIP Check: does your nth term make sense? The 25th term of a sequence that grows by 4 should be about 100 more than the first term.

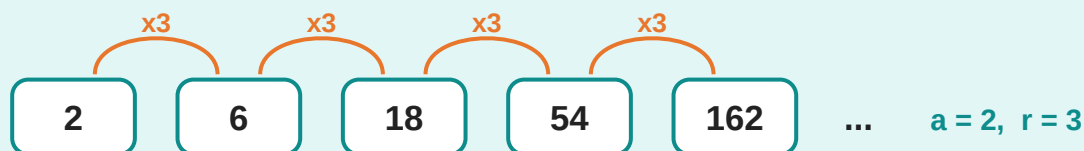
The Multiplier (Geometric)

Multiply by the same number every time

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What it is

A **geometric sequence multiplies** by the same number every time. That number is the **common ratio, r**. The first term is **a**. These grow (or shrink) FAST.



Finding any term

$$\text{nth term} = a \times r^{n-1}$$

$n - 1$ multiplications, just like $n - 1$ jumps. **Example:** 6th term of 2, 6, 18, ... = $2 \times 3^5 = 2 \times 243 = 486$. Check by continuing: $162 \times 3 = 486$.

Adding it up: the geometric series

Sum of the first n terms:

$$\text{Sum} = a \times (r^n - 1) / (r - 1)$$

Example: $2 + 6 + 18 + 54 + 162$ ($n = 5$): $2 \times (3^5 - 1) / (3 - 1) = 2 \times 242 / 2 = 242$. Add them by hand to check: $2 + 6 + 18 + 54 + 162 = 242$. It works!

Shortcut when $r = 2$: the sum is always *one less than the next term*. $1 + 2 + 4 + 8 + 16 = 31 = 32 - 1$.

Shrinking forever: when r is a fraction

If r is between 0 and 1 (like $1/2$), the terms get tinier and tinier, and even an endless series adds up to a number: **Sum forever = $a / (1 - r)$**

Example: $1 + 1/2 + 1/4 + 1/8 + \dots = 1 / (1 - 1/2) = 1 / (1/2) = 2$. Walk half the way to the wall, then half of what's left, forever - you get 2 steps total.

TIP Spot a geometric sequence: divide any term by the one before it. Same answer every time? That's r .

Your Turn: Multipliers

Find r first, then use the formulas

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Name: _____

n th term = $a \times r^{n-1}$ Sum = $a \times (r^n - 1) / (r - 1)$ Sum forever = $a / (1 - r)$ (r a fraction)

- 1 Find the 7th term of 3, 6, 12, 24, ...

$a = \underline{\hspace{1cm}}$ $r = \underline{\hspace{1cm}}$

Answer: _____

- 2 Find the 5th term of 1, 5, 25, 125, ...

$a = \underline{\hspace{1cm}}$ $r = \underline{\hspace{1cm}}$

Answer: _____

- 3 Find the sum: $1 + 2 + 4 + 8 + 16 + 32 + 64$

Try the $r = 2$ shortcut, then check with the formula.

Answer: _____

- 4 Find the sum: $5 + 15 + 45 + 135$

$n = 4$

Answer: _____

- 5 A unicorn's magic doubles every hour. It starts with 3 sparkles. How many sparkles after 10 hours? How many sparkles in TOTAL did it ever have?

A term, then a sum.

Answer: _____

- 6 Find the sum forever: $8 + 4 + 2 + 1 + 1/2 + \dots$

$a = 8, r = 1/2$

Answer: _____

TIP Powers of 2 to know by heart: 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024. Powers of 3: 3, 9, 27, 81, 243, 729.

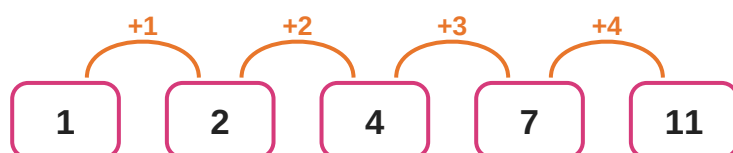
Patterns of Addition

The Growing Jumper: the jumps themselves follow a pattern

What it is

The jumps aren't all the same - but they **follow their own pattern**, usually growing by 1 each time (+1, +2, +3, +4...). To find the next number: figure out the next jump first, then add it.

Example: 1, 2, 4, 7, 11, ...



Gaps: +1, +2, +3, +4. Next gap: **+5**. Next number: $11 + 5 = 16$. Then +6 gives 22, +7 gives 29...

...

Famous growing jumpers

Square numbers: 1, 4, 9, 16, 25, 36

gaps +3, +5, +7, +9... (the odd numbers!)

nth term = $n \times n$

Triangular numbers: 1, 3, 6, 10, 15, 21

gaps +2, +3, +4, +5...

nth term = $n \times (n + 1) / 2$

Handshakes / lines: 0, 1, 3, 6, 10, 15

same gaps as triangular, one step behind

n people shake hands $n \times (n - 1) / 2$ times

Your turn

Find the next two numbers (draw the bridges - what pattern do the jumps make?):



What is the 10th square number? _____ The 10th triangular number? _____

TIP If the jumps grow by 1 each time, the formula has an $n \times n$ in it somewhere. Square and triangular numbers show up in contests ALL the time.

The Ping-Pong (Alternating)

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What it is

The bridges flip back and forth (+10, -11, +10, -11...). That's a sign that **two different sequences are interleaved**: one lives in the odd positions, the other in the even positions. Split them apart and crack each one separately.

Example: 10, 20, 9, 19, 8, 18, ...



Blue team (1st, 3rd, 5th...): 10, 9, 8 - subtract 1.

Pink team (2nd, 4th, 6th...): 20, 19, 18 - subtract 1.

Next is blue's turn: **7**. Then pink: **17**. The 7th term is 7, the 8th is 17.

Finding a far-away term: the 20th term is pink's 10th number (even positions, $20 / 2 = 10$). Pink starts at 20 and subtracts 1 nine times: $20 - 9 = 11$.

Another kind of ping-pong: alternating signs

1, -2, 3, -4, 5, -6, ... The sizes go 1, 2, 3, 4... and the sign flips. Sum of the first 6: $(1 - 2) + (3 - 4) + (5 - 6) = -1 - 1 - 1 = -3$. Pair them up!

Your turn

Split into teams and find the next two:



TIP Odd position -> first team, even position -> second team. The 15th term is the 8th number of the first team: $(15 + 1) / 2 = 8$.

The Fibonacci Sequence

Add the two before to get the next

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The rule

Start with 1, 1. Every number after that is the **sum of the two before it**. Bridges won't crack this one (gaps are 0, 1, 1, 2, 3, 5... the sequence itself!), so look for the add-the-last-two pattern.



Contest tricks with Fibonacci

Sum shortcut: the sum of the first n Fibonacci numbers is **the $(n + 2)$ th number minus 1**.
 $1 + 1 + 2 + 3 + 5 + 8 = 20$, and the 8th number is 21. $21 - 1 = 20$. It works!

Odd/even pattern: odd, odd, even, odd, odd, even... every 3rd Fibonacci number is even.

Working backwards: if you know two neighbors, you can find the one before: $34 - 21 = 13$. **Fib-like sequences** use the same rule with a different start: 2, 5, 7, 12, 19, 31...

Your turn

- Write the next three Fibonacci numbers after 34: _____, _____, _____
- A Fib-like sequence starts 3, 4, ... Write the first 7 terms: 3, 4, _____, _____, _____, _____, _____
- In a Fib-like sequence, two neighbors are 11 and 18. What number came right before 11? _____
- What is $1 + 1 + 2 + 3 + 5 + 8 + 13 + 21$? Use the shortcut: _____

TIP Fibonacci numbers to know: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144. They show up in pinecones, sunflowers, and rabbit problems!

Formula Cheat Sheet

Everything on one page

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STEADY JUMPER (arithmetic): add d each time

$$\text{nth term} = a + d \times (n - 1)$$

$$4, 7, 10, 13 \dots \quad a = 4, d = 3$$

$$\text{Sum} = \text{average of first and last} \times n = (\text{first} + \text{last}) / 2 \times n$$

Fenceposts: from a to b there are $b - a + 1$ numbers

MULTIPLIER (geometric): multiply by r each time

$$\text{nth term} = a \times r^{(n-1)}$$

$$2, 6, 18, 54 \dots \quad a = 2, r = 3$$

$$\text{Sum of } n \text{ terms} = a \times (r^n - 1) / (r - 1)$$

$$\text{Sum forever (r a fraction)} = a / (1 - r)$$

GROWING JUMPER: gaps change steadily

Find the next gap, then add it

$$1, 2, 4, 7, 11 \dots \quad \text{gaps } +1, +2, +3, +4$$

Squares: $n \times n$ (1, 4, 9, 16...)

Triangular: $n \times (n + 1) / 2$ (1, 3, 6, 10...)

PING-PONG: two sequences take turns

Odd positions = team 1, even positions = team 2

$$10, 20, 9, 19, 8, 18 \dots$$

nth term: find which team, then which number on that team

Alternating signs: pair them up to add

FIBONACCI: add the two before

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144

Sum of first $n = (n + 2)\text{th term} - 1$

Every 3rd one is even

First step for EVERY sequence: draw the bridges.

Your Turn: Mixed Challenge

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Which kind of sequence is it? Crack it, then answer

Name: _____

1 2, 5, 8, 11, ... What is the 30th term?

Type: _____

Answer: _____

2 4, 12, 36, 108, ... What is the 6th term?

Type: _____

Answer: _____

3 1, 3, 6, 10, 15, ... What is the 12th term?

Type: _____

Answer: _____

4 7, 50, 9, 45, 11, 40, ... What is the 12th term?

Type: _____

Answer: _____

5 What is $10 + 20 + 30 + \dots + 200$?

Type: _____

Answer: _____

6 1, 1, 2, 3, 5, 8, ... What is the 11th term?

Type: _____

Answer: _____

TIP Name the type first - the color of each box is a hint! Then pick the formula from the cheat sheet.