

Counting with Replacement

First principles: a slot for each choice, multiply the slots

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The slot method

A dragon's lock has **3 dials**, each with digits 0-9. How many codes are possible?

Draw a **slot** for each dial. In each slot write how many choices there are. Each dial has 10 choices, and using a digit on one dial doesn't use it up for the next - that's **with replacement**. Then **multiply**.

$$\begin{array}{|c|} \hline 10 \\ \hline \end{array} \times \begin{array}{|c|} \hline 10 \\ \hline \end{array} \times \begin{array}{|c|} \hline 10 \\ \hline \end{array} = 1,000 \text{ codes}$$

dial 1 dial 2 dial 3

Why multiply? A tree of choices

A monster picks a hat (red or blue) and a scarf (green, yellow or purple). How many outfits?

For EACH hat there are 3 scarves. 2 hats x 3 scarves = **6 outfits**. The tree shows every branch:



More examples:

Flip a coin 4 times: $2 \times 2 \times 2 \times 2 = 16$ outcomes.

A 2-letter spell word from A-Z (letters CAN repeat): $26 \times 26 = 676$.

Lunch: 4 sandwiches, 3 drinks, 2 desserts: $4 \times 3 \times 2 = 24$ lunches.

The key question for every counting problem

Does picking something use it up? If NO (digits on a dial, coin flips), every slot has the same number of choices. If YES (people in line, books on a shelf), the slots shrink - that's the next page.

Your turn

1. A 4-digit lock, digits 0-9: _____
2. A 3-letter password, letters can repeat: _____
3. 5 shirts and 4 pants: how many outfits? _____
4. Roll a die 3 times: how many results? _____

Permutations

Order matters, no replacement: the slots shrink by one each time

Lining up: each pick uses one up

4 dragons line up for a photo. How many different orders?

First spot: 4 choices. One dragon is now used up, so the second spot has only 3 choices. Then 2. Then 1.

$$\begin{array}{cccc} \boxed{4} & \times & \boxed{3} & \times & \boxed{2} & \times & \boxed{1} & = & 24 \text{ orders} \\ 1\text{st} & & 2\text{nd} & & 3\text{rd} & & 4\text{th} & & \end{array}$$

$4 \times 3 \times 2 \times 1$ is written $4!$ and called '4 factorial'. $5! = 120$, $6! = 720$, $3! = 6$, $1! = 1$.

Only some of them: stop filling slots early

7 monsters compete. How many ways to award **gold, silver and bronze**?

Only 3 slots to fill (the other 4 monsters get nothing). Gold: 7 choices. Silver: 6. Bronze: 5. Stop.

$$\begin{array}{ccc} \boxed{7} & \times & \boxed{6} & \times & \boxed{5} & = & 210 \text{ ways} \\ \text{gold} & & \text{silver} & & \text{bronze} & & \end{array}$$

This is called ' $P(7, 3)$ ' or '7 pick 3'. Just start at 7 and count down 3 slots.

Replacement vs. no replacement - side by side

3-digit code, digits CAN repeat: $10 \times 10 \times 10 = 1000$

3-digit code, digits CANNOT repeat: $10 \times 9 \times 8 = 720$

Same slots. The only difference: do the choices shrink? Ask that question first, every time.

Your turn

1. 5 books in a row: _____
2. 1st and 2nd place from 8 runners: _____
3. $6! =$ _____
4. A 4-letter word using the letters W, I, Z, A, R, D with no repeats: _____

Permutations with Duplicates

BOOKKEEPER: when some of the things are identical

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The problem with identical letters

How many ways to arrange the letters of **CAT**? $3! = 6$: CAT, CTA, ACT, ATC, TCA, TAC. Fine.

Now **MOM**. $3! = 6$ says six, but write them out: MOM, MMO, OMM... and the 'other' M gives the same words again. There are really only **3**. The two M's are **identical**, so swapping them doesn't make a new word - we counted every word twice.

The fix: divide out the swaps you can't see

Start with the total as if every letter were different: $n!$. Then for each group of identical letters, **divide by (how many)!** - that removes all the invisible swaps.

$$\text{arrangements} = n! / (a! \times b! \times c! \dots)$$

MOM: 3 letters, two M's $\rightarrow 3! / 2! = 6 / 2 = 3$. **SEES**: 4 letters, two S's and two E's $\rightarrow 4! / (2! \times 2!) = 24 / 4 = 6$.

The famous one: BOOKKEEPER

10 letters. Count the repeats: **O** x2, **K** x2, **E** x3. (B, P, R appear once - $1! = 1$, so they don't matter.)

$$10! / (2! \times 2! \times 3!) = 3,628,800 / 24 = 151,200$$

How to compute it without a calculator melting: cancel first. $10! / 3! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4$ (the $3 \times 2 \times 1$ cancels). Then divide by $2 \times 2 = 4$: $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 / 4 = 10 \times 9 \times 8 \times 7 \times 6 \times 5 = 151,200$.

Why E gets 3!: the three E's can be shuffled $3! = 6$ ways without changing the word, so every word was counted 6 times.

Your turn

1. DAD: _____
2. BALL: _____
3. BOOK: _____
4. LEVEL: _____
5. BANANA (3 A's, 2 N's): _____

Combinations

Order doesn't matter: choosing a group, not a lineup

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Permutation or combination? Does the ORDER make it different?

Gold, silver, bronze for 3 of 7 monsters: Ann-Bob-Cal is different from Bob-Ann-Cal. Order matters → **permutation**: $7 \times 6 \times 5 = 210$.

A team of 3 from 7 monsters: Ann-Bob-Cal is the SAME team as Bob-Ann-Cal. Order doesn't matter → **combination**.

The permutation count is too big - it counted every team once for EACH way to order it. A team of 3 can be ordered $3! = 6$ ways. So:

$$\text{teams} = 210 / 3! = 210 / 6 = 35$$

The formula: 'n choose r'

$$C(n, r) = (n \times (n-1) \times \dots \text{for } r \text{ slots}) / r!$$

Top: fill r slots the permutation way (count down from n). Bottom: divide by r! to erase the order.

$C(7, 3) = (7 \times 6 \times 5) / (3 \times 2 \times 1) = 210 / 6 = 35$. **$C(5, 2) = (5 \times 4) / (2 \times 1) = 10$.** **$C(10, 2) = (10 \times 9) / 2 = 45$.**

Handy facts and the handshake problem

$C(n, 1) = n$ (pick one thing: n ways). **$C(n, n) = 1$** (take everything: one way). **$C(n, r) = C(n, n-r)$** : choosing 3 to be ON the team is the same as choosing 4 to be OFF it, so $C(7, 3) = C(7, 4) = 35$.

Handshakes: 8 goblins each shake hands with every other goblin once. How many handshakes? A handshake is a pair of goblins and order doesn't matter: $C(8, 2) = (8 \times 7) / 2 = 28$. (This is the $n(n-1)/2$ formula from the sequences guide!)

Lines through points: 6 dots, no three in a row. How many lines connect two dots? $C(6, 2) = 15$.

Your turn

1. $C(6, 2) =$ _____
2. $C(8, 3) =$ _____
3. Pick 2 toppings from 7: _____
4. 12 dragons each shake hands with every other dragon once: _____

Bookshelf Problems

9 books, 4 of them Harry Potter: arranging with rules

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Start simple: 9 books, any order

9 slots, each pick uses up a book: $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 9! = 362,880$ ways. (A permutation.)

The main event: the 4 Harry Potter books must be all together

Step 1 - glue them. Tape the 4 Harry Potter books into one fat super-book. Now there are 5 other books + 1 super-book = **6 things** to arrange on the shelf: $6! = 720$ ways.



5 single books + 1 super-book = 6 things $\rightarrow 6! = 720$

Step 2 - un-glue. Inside the super-book, the 4 Harry Potter books can be in any order: $4! = 24$ ways. Every one of the 720 shelf arrangements comes in 24 versions.

Step 3 - multiply. $720 \times 24 = 17,280$ ways.

$$(\text{things to arrange})! \times (\text{books inside the glue})! = 6! \times 4! = 17,280$$

Twists on the same shelf

HP books together AND in series order (1-2-3-4): the inside has only 1 order, so just $6! = 720$.

HP books together at the far left: super-book fixed in slot 1, other 5 free: $5! \times 4! = 120 \times 24 = 2,880$.

HP books NOT all together: all minus together = $362,880 - 17,280 = 345,600$.

Other rules you'll see

One book fixed in a spot: place it first (1 way), arrange the rest. **Only k of them fit:** fill k slots counting down. **Which k are chosen, order ignored:** $C(n, k)$.

TIP Glue, arrange, un-glue: (number of things)! $\times$ (books inside)! . 'Not together' = all minus together.

Your Turn: Advanced Counting

Replacement? Order? Duplicates? Decide, then count

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Name: _____

- 1 A dragon's vault has a 5-digit code. Digits can repeat. How many codes?

Answer: _____

- 2 6 monsters line up for a race photo. How many orders?

Answer: _____

- 3 How many ways to arrange the letters of GOBLIN? OF MUMMY?

Answer: _____

- 4 A coven of 9 witches picks a committee of 3. How many committees?

Answer: _____

- 5 A wizard picks a wand (5 kinds) and a hat (3 kinds) and a cloak (4 kinds). How many outfits?

Answer: _____

- 6 8 books on a shelf, and 3 of them are a trilogy that must stay together. How many ways?

Answer: _____

- 7 10 trolls shake hands with each other once. How many handshakes?

Answer: _____

- 8 6 books on a shelf. 2 of them must stay together AND be in a fixed order. How many ways?

Answer: _____

TIP Colors: blue = replacement, green = permutation, purple = duplicates, orange = combination, pink = bookshelf.

Advanced Counting Cheat Sheet

Ask the questions in order

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WITH REPLACEMENT (choices don't get used up)

Slot for each pick, same number in every slot, multiply

3-digit code: $10 \times 10 \times 10$

PERMUTATION (order matters, no replacement)

Slots shrink: $n \times (n-1) \times (n-2) \dots$ for as many slots as you fill

All of them in a row: $n!$

$7 \times 6 \times 5 = 210$ $5! = 120$

DUPLICATES (some things identical)

$n!$ divided by (count)! for each repeated thing

BOOKKEEPER: $10! / (2! 2! 3!)$

COMBINATION (order doesn't matter)

Permutation count divided by $r!$ to erase the order

$C(n,2) = n(n-1)/2$ is the handshake number

$C(7,3) = 7 \times 6 \times 5 / 3! = 35$

BOOKSHELF RULES

Fixed spot: place it first (1 choice), arrange the rest

Together: glue into one thing, then x ways to order inside

'Not': all minus 'yes'

1) Does a pick get used up? 2) Does order matter? 3) Are any of the things identical?