

Volume Practice Problems

AB Calculus Review — Solids of Revolution & Solids with Known Cross-Sections

Name: _____ Date: _____

Formulas to remember

Rotation about a horizontal axis (integrate in x): $V = \pi \int [R(x)^2 - r(x)^2] dx$

Rotation about a vertical axis (integrate in y): $V = \pi \int [R(y)^2 - r(y)^2] dy$

The radius is always the **distance from the axis of rotation to the curve** — watch for shifted axes like $y = 4$ or $x = 2$!

Known cross-sections (no rotation): $V = \int A(x) dx$, where s = distance between the curves:

Cross-section shape	Area in terms of s
Square (side s)	s^2
Semicircle (diameter s)	$(\pi/8)s^2$
Equilateral triangle (side s)	$(\sqrt{3}/4)s^2$
Isosceles right triangle (leg s)	$(1/2)s^2$

Trap to avoid: for semicircles, s is the **diameter**, so the radius is $s/2$ and $A = \frac{1}{2}\pi(s/2)^2 = (\pi/8)s^2$.

Section A — Solids of Revolution (Disk & Washer)

Part 1 — Warm-ups (single disks)

- A1.** Let R be the region bounded by $y = \sqrt{x}$, the x -axis, and the line $x = 4$. Find the volume of the solid formed when R is rotated about the **x -axis**.
- A2.** Let R be the region bounded by $y = x^2$, the y -axis, and the line $y = 4$ (for $x \geq 0$). Find the volume of the solid formed when R is rotated about the **y -axis**.
- A3.** Let R be the region bounded by $y = 1/x$, the x -axis, $x = 1$, and $x = 2$. Find the volume of the solid formed when R is rotated about the **x -axis**.

Part 2 — Washers (region between two curves)

- A4.** Let R be the region bounded by $y = x$ and $y = x^2$. Find the volume of the solid formed when R is rotated about the **x -axis**.
- A5.** Let R be the region bounded by $y = 2x$ and $y = x^2$. Find the volume of the solid formed when R is rotated about the **y -axis**. (Hint: solve each equation for x first.)
- A6.** Let R be the region bounded by $y = e^x$, the line $y = e$, and the y -axis. Find the volume of the solid formed when R is rotated about the **x -axis**.

Part 3 — Rotation about lines other than the axes

- A7.** Let R be the region bounded by $y = x^2$ and the line $y = 4$. Find the volume of the solid formed when R is rotated about the line $y = 4$.
- A8.** Let R be the region bounded by $y = x^2$, the x -axis, and the line $x = 1$. Find the volume of the solid formed when R is rotated about the line $x = 2$.
- A9.** Let R be the region bounded by $y = \sqrt{x}$ and $y = x/2$. Find the volume of the solid formed when R is rotated about the line $y = -1$.

Challenge

- A10.** Let R be the region bounded by $y = \sin x$ and the x -axis on $[0, \pi]$. Find the volume of the solid formed when R is rotated about the x -axis. (You'll need the identity $\sin^2 x = (1 - \cos 2x)/2$.)

Section B — Solids with Known Cross-Sections

Part 1 — Warm-ups

- B1.** The base of a solid is the region bounded by $y = \sqrt{x}$, the x -axis, and $x = 4$. Cross-sections perpendicular to the **x -axis** are **squares**. Find the volume.
- B2.** The base of a solid is the region bounded by $y = x^2$ ($x \geq 0$), the y -axis, and the line $y = 4$. Cross-sections perpendicular to the **y -axis** are **squares**. Find the volume.
- B3.** The base of a solid is the region bounded by $y = 4 - x^2$ and the x -axis. Cross-sections perpendicular to the **x -axis** are **semicircles** (with diameter in the base). Find the volume.

Part 2 — Two curves

- B4.** The base of a solid is the region bounded by $y = x$ and $y = x^2$. Cross-sections perpendicular to the **x -axis** are **semicircles** with diameter running between the two curves. Find the volume.
- B5.** The base of a solid is the region bounded by $y = x$ and $y = x^2$. Cross-sections perpendicular to the **x -axis** are **isosceles right triangles** with a leg in the base. Find the volume.
- B6.** The base of a solid is the region bounded by $y = \sqrt{x}$ and $y = x/2$. Cross-sections perpendicular to the **x -axis** are **equilateral triangles**. Find the volume.

Part 3 — Harder integrands

- B7.** The base of a solid is the region bounded by $y = e^x$, $y = 1$, and $x = 1$. Cross-sections perpendicular to the **x -axis** are **semicircles** with diameter running between the two curves. Find the volume.
- B8. (Challenge)** The base of a solid is the region bounded by $y = \cos x$ and the x -axis on $[-\pi/2, \pi/2]$. Cross-sections perpendicular to the **x -axis** are **squares**. Find the volume.

Key habits

- No π appears unless the shape of the slice has a circle in it — don't reflexively write πf out of rotation habit.
- s is always **top curve minus bottom curve** (or right minus left, if slicing in y).
- It's $R^2 - r^2$, never $(R - r)^2$ — square each radius before subtracting.
- Rotating about a vertical axis? Everything must be in terms of y — rewrite the curves as $x = f(y)$ and use y -limits.

Answer Key — Section A (with full setups)

A1. Disk, radius $R(x) = \sqrt{x}$: $V = \pi \int_0^4 (\sqrt{x})^2 dx = \pi \int_0^4 x dx = \pi [x^2/2]_0^4 = 8\pi$

A2. Rotating about a vertical axis $\rightarrow$ integrate in y . Solve $y = x^2$ for x : $x = \sqrt{y}$. Disk radius $R(y) = \sqrt{y}$: $V = \pi \int_0^4 (\sqrt{y})^2 dy = \pi \int_0^4 y dy = 8\pi$

A3. Disk, $R(x) = 1/x$: $V = \pi \int_1^2 (1/x)^2 dx = \pi [-1/x]_1^2 = \pi(-1/2 + 1) = \pi/2$

A4. Curves intersect at $x = 0$ and $x = 1$. On $[0, 1]$, $y = x$ is on top. Outer radius x , inner radius x^2 :
 $V = \pi \int_0^1 (x^2 - x^4) dx = \pi [x^3/3 - x^5/5]_0^1 = \pi(1/3 - 1/5) = 2\pi/15$

A5. Intersect at $(0, 0)$ and $(2, 4)$, so y goes 0 to 4. Solving for x : the parabola gives $x = \sqrt{y}$ (farther from the y -axis $\rightarrow$ outer), the line gives $x = y/2$ (inner): $V = \pi \int_0^4 [(\sqrt{y})^2 - (y/2)^2] dy = \pi \int_0^4 (y - y^2/4) dy = \pi [y^2/2 - y^3/12]_0^4 = \pi(8 - 16/3) = 8\pi/3$

A6. The region runs from $x = 0$ to $x = 1$ (where $e^x = e$). Outer radius e (the line), inner radius e^x :
 $V = \pi \int_0^1 (e^2 - e^{2x}) dx = \pi [e^2x - e^{2x}/2]_0^1 = \pi(e^2 - e^2/2 + 1/2) = \pi(e^2 + 1)/2$

A7. Curves intersect at $x = \pm 2$. Distance from the axis $y = 4$ down to the parabola: $R(x) = 4 - x^2$. Single disk (the region touches the axis): $V = \pi \int_{-2}^2 (4 - x^2)^2 dx = \pi \int_{-2}^2 (16 - 8x^2 + x^4) dx = \pi [16x - 8x^3/3 + x^5/5]_{-2}^2 = 512\pi/15$

A8. Vertical axis $\rightarrow$ integrate in y from 0 to 1. For a horizontal slice at height y , the region runs from $x = \sqrt{y}$ to $x = 1$. Distances from the axis $x = 2$: outer $R = 2 - \sqrt{y}$, inner $r = 2 - 1 = 1$: $V = \pi \int_0^1 [(2 - \sqrt{y})^2 - 1] dy = \pi \int_0^1 (3 - 4\sqrt{y} + y) dy = \pi [3y - (8/3)y^{3/2} + y^2/2]_0^1 = \pi(3 - 8/3 + 1/2) = 5\pi/6$

A9. Curves intersect at $x = 0$ and $x = 4$; $\sqrt{x}$ is on top. Shifting the axis down to $y = -1$ adds 1 to each radius: outer $R = \sqrt{x} + 1$, inner $r = x/2 + 1$: $V = \pi \int_0^4 [(\sqrt{x} + 1)^2 - (x/2 + 1)^2] dx = \pi \int_0^4 (2\sqrt{x} - x^2/4) dx = \pi [(4/3)x^{3/2} - x^3/12]_0^4 = \pi(32/3 - 16/3) = 16\pi/3$

A10. Disk, $R(x) = \sin x$: $V = \pi \int_0^\pi \sin^2 x dx = \pi \int_0^\pi (1 - \cos 2x)/2 dx = \pi [x/2 - (\sin 2x)/4]_0^\pi = \pi^2/2$

Answer Key — Section B (with full setups)

B1. $s = \sqrt{x}$, so $A(x) = (\sqrt{x})^2 = x$: $V = \int_0^4 x \, dx = \mathbf{8}$

B2. Perpendicular to the y-axis $\rightarrow$ slice in y. Side runs from the y-axis to the parabola: $s = \sqrt{y}$, so $A(y) = y$: $V = \int_0^4 y \, dy = \mathbf{8}$

B3. $s = 4 - x^2$ (diameter), $A(x) = (\pi/8)(4 - x^2)^2$: $V = (\pi/8) \int_{-2}^2 (16 - 8x^2 + x^4) \, dx = (\pi/8)(512/15) = \mathbf{64\pi/15}$

B4. Curves meet at $x = 0, 1$ with $y = x$ on top: $s = x - x^2$, $A(x) = (\pi/8)(x - x^2)^2$: $\int_0^1 (x^2 - 2x^3 + x^4) \, dx = 1/3 - 1/2 + 1/5 = 1/30$, so $V = (\pi/8)(1/30) = \mathbf{\pi/240}$

B5. Same $s = x - x^2$, but $A(x) = (1/2)(x - x^2)^2$: $V = (1/2)(1/30) = \mathbf{1/60}$ (Same base and same integral as B4 — only the area formula changed.)

B6. Curves meet at $x = 0, 4$ with $y = \sqrt{x}$ on top: $s = \sqrt{x} - x/2$, $s^2 = x - x^{3/2} + x^2/4$: $\int_0^4 (x - x^{3/2} + x^2/4) \, dx = [x^2/2 - (2/5)x^{5/2} + x^3/12]_0^4 = 8 - 64/5 + 16/3 = 8/15$; $V = (\sqrt{3}/4)(8/15) = \mathbf{2\sqrt{3}/15}$

B7. The region runs from $x = 0$ (where $e^x = 1$) to $x = 1$: $s = e^x - 1$, $A(x) = (\pi/8)(e^x - 1)^2$: $\int_0^1 (e^{2x} - 2e^x + 1) \, dx = [e^{2x}/2 - 2e^x + x]_0^1 = (e^2/2 - 2e + 1) - (1/2 - 2) = e^2/2 - 2e + 5/2$; $V = (\pi/8)(e^2/2 - 2e + 5/2) = \mathbf{(\pi/16)(e^2 - 4e + 5)}$

B8. $s = \cos x$, $A(x) = \cos^2 x$. Use $\cos^2 x = (1 + \cos 2x)/2$: $V = \int_{-\pi/2}^{\pi/2} (1 + \cos 2x)/2 \, dx = [x/2 + (\sin 2x)/4]_{-\pi/2}^{\pi/2} = \mathbf{\pi/2}$