

AP Calculus AB Study Guide: Integration by U-Substitution

1. The Big Idea

U-substitution is the **chain rule in reverse**. When an integrand contains a function *and* (some multiple of) its derivative, substitution converts the integral into a simpler one.

$$\int f(g(x)) g'(x) dx = \int f(u) du \quad \text{where } u = g(x)$$

When to try it: Look for a “function inside a function” (composite function) where the derivative of the *inside* function also appears in the integrand — possibly off by a constant factor.

2. The Procedure

1. **Choose u** — usually the “inside” function: the expression under a radical, in an exponent, inside parentheses raised to a power, in a denominator, or inside a trig function.
2. **Compute du** — differentiate: $du = u'(x) dx$.
3. **Solve for dx** (or match up a constant): if $du = 3x^2 dx$, then $x^2 dx = \frac{1}{3} du$.
4. **Rewrite the integral entirely in u** — no x’s should remain!
5. **Integrate** with basic rules.
6. **Back-substitute** to return to x (indefinite integrals only).
7. **Definite integrals:** change the limits to u-values using $u = g(x)$ — then you never return to x.

AP Exam Tip: For definite integrals, changing the bounds is faster and avoids the most common error (using x-bounds with a u-antiderivative).

3. Common Patterns to Recognize

Pattern	Substitution	Result
$\int (ax + b)^n dx$	$u = ax + b$	$\frac{(ax + b)^{n+1}}{a(n+1)} + C$
$\int x e^{x^2} dx$	$u = x^2$	$\frac{1}{2} e^{x^2} + C$
$\int \frac{g'(x)}{g(x)} dx$	$u = g(x)$	$\ln \ g(x)\ + C$
$\int \sin^n x \cos x dx$	$u = \sin x$	$\frac{\sin^{n+1} x}{n+1} + C$
$\int \frac{dx}{\sqrt{a^2 - x^2}}$	$u = \frac{x}{a}$	$\arcsin\left(\frac{x}{a}\right) + C$
$\int \tan x dx$	$u = \cos x$	$-\ln \ \cos x\ + C$

Key derivative facts you’ll need: $-\frac{d}{dx}[\ln x] = \frac{1}{x}$, $\frac{d}{dx}[e^x] = e^x$ - $\frac{d}{dx}[\sin x] = \cos x$, $\frac{d}{dx}[\cos x] = -\sin x$ - $\frac{d}{dx}[\tan x] = \sec^2 x$, $\frac{d}{dx}[\sec x] = \sec x \tan x$

4. Worked Examples

Example A (basic power)

$$\int 2x(x^2 + 5)^4 dx$$

Let $u = x^2 + 5$, so $du = 2x dx$.

$$\int u^4 du = \frac{u^5}{5} + C = \boxed{\frac{(x^2 + 5)^5}{5} + C}$$

Example B (adjusting a constant)

$$\int x^2 \sqrt{x^3 + 1} dx$$

Let $u = x^3 + 1$, so $du = 3x^2 dx \Rightarrow x^2 dx = \frac{1}{3} du$.

$$\frac{1}{3} \int u^{1/2} du = \frac{1}{3} \cdot \frac{2}{3} u^{3/2} + C = \boxed{\frac{2}{9} (x^3 + 1)^{3/2} + C}$$

Example C (definite integral — change the bounds)

$$\int_0^{\pi/4} \tan x \sec^2 x dx$$

Let $u = \tan x$, $du = \sec^2 x dx$. Bounds: $x = 0 \Rightarrow u = 0$; $x = \pi/4 \Rightarrow u = 1$.

$$\int_0^1 u du = \frac{u^2}{2} \Big|_0^1 = \boxed{\frac{1}{2}}$$

Example D (ln pattern)

$$\int \frac{x}{x^2 + 9} dx$$

Let $u = x^2 + 9$, $du = 2x dx$.

$$\frac{1}{2} \int \frac{du}{u} = \boxed{\frac{1}{2} \ln(x^2 + 9) + C}$$

(No absolute value needed — $x^2 + 9 > 0$ always.)

5. Practice Problems

Part I — Fundamentals (indefinite)

1. $\int (3x - 7)^5 dx$

2. $\int x(x^2 - 4)^6 dx$

3. $\int \frac{x^3}{\sqrt{x^4 + 2}} dx$

4. $\int \cos(5x) dx$

5. $\int x e^{-x^2} dx$

6. $\int \frac{(\ln x)^3}{x} dx$

Part II — Intermediate

7. $\int \sin^4 x \cos x dx$

8. $\int \frac{e^x}{1 + e^x} dx$

9. $\int \frac{x + 3}{x^2 + 6x - 1} dx$

10. $\int \sec^2(2x) \tan(2x) dx$

11. $\int \frac{x}{\sqrt{x+1}} dx$ (*hint: after substituting, rewrite x in terms of u*)

12. $\int \frac{dx}{x \ln x}$

Part III — Definite Integrals (change your bounds!)

13. $\int_0^2 x \sqrt{4 - x^2} dx$

14. $\int_1^e \frac{\ln x}{x} dx$

15. $\int_0^{\pi/2} \cos x e^{\sin x} dx$

16. $\int_0^1 \frac{x^2}{1 + x^3} dx$

Part IV — AP-Style Challenge

17. $\int_0^{\pi/3} \frac{\sin x}{\cos^2 x} dx$
18. If $\int_0^4 f(x) dx = 10$, find $\int_0^2 f(2x) dx$.
19. $\int x^3 \sqrt{x^2 + 1} dx$ (*hint: write $x^3 = x^2 \cdot x$ and express x^2 in terms of u*)
20. (Multiple choice, AP-style) $\int_1^2 \frac{6x}{(x^2 + 1)^2} dx = ?$ (A) $\frac{3}{10}$ (B) $\frac{9}{10}$ (C) $\frac{3}{2}$ (D) $\frac{6}{5}$
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6. Top Mistakes to Avoid

1. **Leaving x's in a u-integral.** Every x (including dx) must be converted before integrating. If an x won't cancel, either express it in terms of u (Problems 11, 19) or your choice of u is wrong.
2. **Forgetting to adjust for constants.** If $du = 3x^2 dx$ but your integrand has $x^2 dx$, you must multiply by $\frac{1}{3}$.
3. **Using x-bounds with u-antiderivatives** on definite integrals. Change the bounds when you substitute.
4. **Dropping the +C** on indefinite integrals — a guaranteed lost point on FRQs.
5. **Forgetting the negative** when $u = \cos x$ (since $du = -\sin x dx$).
6. **You cannot pull a variable out of the integral.** Only constants can move outside the integral sign.