

Parabolas: Focus and Directrix

Study Guide and Practice Packet - No Answer Key

Goal: Move fluently between the geometric definition of a parabola and its equation. Be able to find the vertex, focus, directrix, axis of symmetry, direction of opening, focal width, and equation - including after completing the square.

1. The Geometric Definition of a Parabola

A parabola is the set of all points in a plane that are the same distance from a fixed point, called the focus, and a fixed line, called the directrix.

Focus	A fixed point inside the bowl of the parabola.
Directrix	A fixed line outside the parabola.
Vertex	The point on the parabola halfway between the focus and the directrix.
Axis of symmetry	The line through the vertex and focus, perpendicular to the directrix.
p	The directed distance from the vertex to the focus. Its sign determines the direction of opening.

The defining distance relationship is: distance from any point P on the parabola to the focus = perpendicular distance from P to the directrix.

Practice

1. A parabola has focus (0, 3) and directrix $y = -3$. Identify the vertex, axis of symmetry, and direction of opening.
2. A parabola has focus (-4, 1) and directrix $x = 2$. Identify the vertex, axis of symmetry, and direction of opening.
3. The vertex of a parabola is (2, -1), and its focus is (2, 5). Find the directrix.
4. Explain geometrically why the vertex must lie halfway between the focus and the directrix.

2. Standard Forms: Vertical and Horizontal Parabolas

Unlike the familiar function form $y = ax^2 + bx + c$, focus/directrix problems are most naturally handled using standard conic form.

Vertical axis

$$(x - h)^2 = 4p(y - k)$$

Vertex	(h, k)
Focus	(h, k + p)
Directrix	$y = k - p$
Axis	$x = h$
Opening	Up if $p > 0$; down if $p < 0$

Horizontal axis

$$(y - k)^2 = 4p(x - h)$$

Vertex	(h, k)
Focus	$(h + p, k)$
Directrix	$x = h - p$
Axis	$y = k$
Opening	Right if $p > 0$; left if $p < 0$

Important recognition rule: the variable that is squared tells you which direction the parabola does NOT extend. If x is squared, the parabola opens vertically. If y is squared, it opens horizontally.

Practice

1. For $(x - 3)^2 = 12(y + 1)$, identify h , k , p , the vertex, focus, directrix, axis of symmetry, and direction of opening.
2. For $(y + 4)^2 = -20(x - 2)$, identify h , k , p , the vertex, focus, directrix, axis of symmetry, and direction of opening.
3. For $x^2 = -16y$, find the vertex, focus, directrix, and axis.
4. For $(y - 1)^2 = 6(x + 5)$, find the vertex, focus, directrix, and axis.

3. The Meaning of $4p$ and Focal Width

In standard form, the coefficient multiplying the unsquared expression is $4p$ - not p . A common placement-test error is to read p directly from the coefficient.

$$\text{coefficient} = 4p \quad \text{so} \quad p = \text{coefficient} / 4$$

The latus rectum (often called the focal chord) is the chord through the focus perpendicular to the axis of symmetry. Its length is the focal width:

$$\text{focal width} = |4p|$$

For a vertical parabola, the latus rectum endpoints are $(h - 2|p|, k + p)$ and $(h + 2|p|, k + p)$. For a horizontal parabola, they lie vertically above and below the focus.

Practice

1. For $(x + 2)^2 = 24(y - 3)$, find p and the focal width.
2. For $(y - 5)^2 = -8(x + 1)$, find p and the focal width.
3. A vertical parabola has vertex $(1, 2)$ and focal width 20. Give the two possible locations of its focus.
4. A parabola has vertex at the origin and focus $(0, -7)$. Find the equation and focal width.

4. Writing the Equation from Vertex and Focus

Strategy: identify whether the focus is above/below or left/right of the vertex. That determines the orientation. Then compute p as the directed distance from the vertex to the focus and substitute into standard form.

Practice

1. Write the equation of the parabola with vertex $(2, 3)$ and focus $(2, 7)$.

2. Write the equation of the parabola with vertex $(-1, 4)$ and focus $(-6, 4)$.
3. Write the equation of the parabola with vertex $(0, 0)$ and focus $(0, -5/2)$.
4. Write the equation of the parabola whose vertex is $(3, -2)$ and whose focus is $(9, -2)$.
5. A parabola has vertex (a, b) and focus $(a, b + 6)$. Write its equation in terms of a and b .

5. Writing the Equation from Vertex and Directrix

The vertex-to-directrix distance has magnitude $|p|$, but the directrix is on the opposite side of the vertex from the focus. Use the directrix to infer both the orientation and the sign of p .

Practice

1. Write the equation of the parabola with vertex $(1, -2)$ and directrix $y = -6$.
2. Write the equation of the parabola with vertex $(-3, 5)$ and directrix $x = 1$.
3. Write the equation of the parabola with vertex $(0, 0)$ and directrix $y = 3$.
4. Write the equation of the parabola with vertex $(4, -1)$ and directrix $x = -2$.
5. A horizontal parabola has vertex (h, k) and directrix $x = h + 8$. Write its equation.

6. Finding the Equation from Focus and Directrix

If the vertex is not given, first locate it halfway between the focus and the directrix along the perpendicular axis. Then determine p .

Practice

1. Find the equation of the parabola with focus $(0, 4)$ and directrix $y = -4$.
2. Find the equation of the parabola with focus $(5, 2)$ and directrix $x = -1$.
3. Find the equation of the parabola with focus $(-3, -6)$ and directrix $y = 2$.
4. Find the equation of the parabola with focus $(7, -1)$ and directrix $x = 3$.
5. The focus is $(a, b + 10)$ and the directrix is $y = b - 10$. Find the vertex and equation.

7. Converting an Equation to Focus-Directrix Form

An equation may not initially look like $(x - h)^2 = 4p(y - k)$ or $(y - k)^2 = 4p(x - h)$. Complete the square on the squared-variable expression first.

Example structure

$$x^2 - 6x = 8y + 7$$

Complete the square in x , rewrite the equation in standard form, then read h , k , and p . The same idea works when y is the squared variable.

Practice

1. Rewrite $x^2 - 8x - 12y + 4 = 0$ in standard form. Then find the vertex, focus, directrix, and axis of symmetry.
2. Rewrite $y^2 + 6y + 16x - 7 = 0$ in standard form. Then find the vertex, focus, directrix, and axis.
3. Rewrite $2x^2 + 12x - 8y + 10 = 0$ in standard form. Then identify all geometric features.

- Rewrite $3y^2 - 12y - 24x + 9 = 0$ in standard form. Then identify all geometric features.
- A parabola is given by $x^2 + 4x + 20y + 24 = 0$. Determine whether it opens up or down before completing the square, then verify your prediction.

8. Connecting Focus-Directrix Form to $y = a(x - h)^2 + k$

For a vertical parabola:

$$(x - h)^2 = 4p(y - k)$$

Solving for y gives:

$$y - k = [1/(4p)](x - h)^2$$

$$y = a(x - h)^2 + k, \quad \text{where} \quad a = 1/(4p)$$

Thus p and a are reciprocally related. A small $|p|$ gives a larger $|a|$ and a narrower parabola; a large $|p|$ gives a smaller $|a|$ and a wider parabola.

Practice

- For $y = (1/12)(x - 2)^2 - 4$, find p , the focus, and the directrix.
- For $y = -(1/20)(x + 3)^2 + 1$, find p , the focus, and the directrix.
- A vertical parabola has vertex at the origin and passes through $(6, 3)$. Find its focus and directrix.
- Which is wider: $y = (1/8)x^2$ or $y = (1/20)x^2$? Explain using p , not just a .
- Find a in $y = a(x - 4)^2 + 2$ if the focus is $(4, -1)$.

9. Graphing from Focus and Directrix

A strong sketch should show more than the vertex. Use the focus, directrix, axis of symmetry, and focal width to establish scale and orientation.

Suggested graphing process

- Plot the vertex.
- Plot the focus.
- Draw the directrix as a dashed line.
- Draw the axis of symmetry.
- Use the latus rectum endpoints or another pair of symmetric points.
- Sketch a smooth curve symmetric about the axis.

Practice

- Sketch $(x - 1)^2 = 8(y + 2)$. Label the vertex, focus, directrix, axis, and latus rectum endpoints.
- Sketch $(y + 3)^2 = -12(x - 4)$. Label all key features.
- Sketch $x^2 = 4y$ and $x^2 = 16y$ on the same axes. Explain geometrically why one is wider.
- Sketch two parabolas with the same vertex and axis but opposite values of p . Describe what stays the same and what changes.

10. Using the Distance Definition Directly

For a point $P(x, y)$ on a parabola, its distance to the focus equals its perpendicular distance to the directrix. A difficult test may ask you to use this definition rather than a memorized formula.

Practice

1. The focus is $(0, 2)$ and the directrix is $y = -2$. Starting from the distance formula, derive the equation of the parabola.
2. The focus is $(3, 0)$ and the directrix is $x = -3$. Starting from the distance definition, derive the equation.
3. Show that the point $(4, 2)$ lies on the parabola with focus $(0, 2)$ and directrix $x = 8$ by comparing distances.
4. A point (x, y) is equidistant from focus $(0, p)$ and directrix $y = -p$. Derive $x^2 = 4py$.

11. Problem-Solving and Applications

Parabolic reflectors use the focus property: rays traveling parallel to the axis reflect toward the focus. Algebra II problems may model satellite dishes, headlights, microphones, or arches using parabolas.

Practice

1. A satellite dish has a parabolic cross-section 12 ft wide and 3 ft deep. Place the vertex at the origin and the dish opening upward. Find the location of the focus.
2. A parabolic reflector is 20 cm wide and 5 cm deep. Find how far the receiver should be placed from the vertex.
3. A parabolic arch is 24 ft wide at ground level and 18 ft high at its center. Put the vertex at the top of the arch. Write an equation for the parabola and locate its focus.
4. A flashlight reflector has vertex at the origin and focus 1.5 cm from the vertex. Write an equation for its cross-section if it opens to the right.

12. Mixed Placement-Test Challenge

These problems do not tell you which method to use.

1. Find the focus and directrix of $(x + 5)^2 = -28(y - 2)$.
2. Write the equation of a parabola with focus $(4, -1)$ and directrix $x = -2$.
3. Rewrite $y^2 - 10y - 12x + 1 = 0$ in standard form and identify all key features.
4. A parabola has vertex $(-2, 3)$, opens right, and has focal width 16. Write its equation.
5. A parabola has directrix $y = 7$ and vertex $(1, 2)$. Find its focus and equation.
6. For $y = -(1/16)(x - 6)^2 + 5$, find the focus and directrix.
7. The focus is $(-3, 8)$ and directrix is $y = -4$. Find the vertex, p , equation, and focal width.
8. A vertical parabola passes through $(8, 4)$, has vertex at the origin, and opens upward. Find its focus.
9. Determine whether the point $(6, 3)$ lies on the parabola whose focus is $(0, 3)$ and directrix is $y = -3$. Justify using distances.

10. Two parabolas have the same vertex. One has $p = 2$ and the other $p = 8$. Compare their focus locations, directrices, and widths.

Mastery Checklist

- ☐ I can explain the focus-directrix definition of a parabola.
- ☐ I can distinguish vertical and horizontal parabolas immediately from standard form.
- ☐ I know both standard forms and understand the role of h , k , and p .
- ☐ I remember that the coefficient is $4p$, not p .
- ☐ I can find the vertex, focus, directrix, axis of symmetry, and direction of opening.
- ☐ I can find the focal width and latus rectum endpoints.
- ☐ I can write an equation from a vertex and focus.
- ☐ I can write an equation from a vertex and directrix.
- ☐ I can find the vertex and equation when only the focus and directrix are given.
- ☐ I can complete the square to convert a general equation to focus-directrix standard form.
- ☐ I can connect $y = a(x - h)^2 + k$ with p using $a = 1/(4p)$.
- ☐ I can sketch a parabola accurately using its geometric features.
- ☐ I can derive a parabola equation from the distance definition.
- ☐ I can solve application problems involving parabolic reflectors or arches.
- ☐ I can solve mixed problems without being told which formula or method to use.