

Matrices Study Guide

Placement-Test Review & Practice

Goal

Recognize, set up, and solve advanced Algebra II matrix problems accurately and efficiently. Be especially comfortable with matrix multiplication, determinants, inverses, systems, and row reduction.

Recommended approach

- Work each practice set without notes first.
- Write matrix dimensions beside multiplication problems before multiplying.
- Show row-by-column work rather than trying to do matrix multiplication mentally.
- For systems, be able to switch comfortably among elimination, inverse matrices, and row reduction.
- Mark any problem that takes more than about 2-3 minutes for targeted review.

Contents

- 1. Matrix Vocabulary, Dimensions, and Equality
- 2. Matrix Addition, Subtraction, and Scalar Multiplication
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1. Matrix Vocabulary, Dimensions, and Equality

Know

- A matrix is a rectangular array of numbers arranged in rows and columns.
- The dimensions of a matrix are written rows x columns. A 3 x 2 matrix has 3 rows and 2 columns.
- An entry a_{ij} is the entry in row i , column j .

- Two matrices are equal only if they have the same dimensions and every corresponding entry is equal.
- Common types include row matrices, column matrices, square matrices, diagonal matrices, identity matrices, and zero matrices.

$$A = \begin{bmatrix} 4 & -2 & 7 \\ 0 & 5 & 1 \end{bmatrix}$$

1. State the dimensions of A and identify a_{23} .

$$A = \begin{bmatrix} 3 & -1 & 8 & 0 \\ 5 & 6 & -2 & 4 \\ 7 & 9 & 1 & -3 \end{bmatrix}$$

2. Find x and y if the matrices are equal.

$$\begin{bmatrix} x+2 & 5 \\ 3 & 2y-1 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 3 & 11 \end{bmatrix}$$

3. Classify each matrix as square, row, column, diagonal, identity, or zero when applicable.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 4 & -2 & 7 \end{bmatrix} \quad R = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

2. Matrix Addition, Subtraction, and Scalar Multiplication

Know

- Matrices can be added or subtracted only when they have exactly the same dimensions.
- Add or subtract corresponding entries.
- Scalar multiplication means multiplying every entry of a matrix by the same number.
- Distribute scalars through matrix sums just as in ordinary algebra: $k(A + B) = kA + kB$.

1. Find $A + B$ and $A - B$.

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 6 \end{bmatrix} \quad B = \begin{bmatrix} -5 & 4 \\ 7 & 0 \end{bmatrix}$$

2. Evaluate $3A - 2B$.

$$A = \begin{bmatrix} 1 & 4 & -2 \\ 3 & 0 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -1 & 5 \\ 2 & 4 & -3 \end{bmatrix}$$

3. Solve for X.

$$2X + \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ -2 & 10 \end{bmatrix}$$

4. Explain why the following sum is undefined.

A is 2×3 and B is 3×2 . Find $A + B$.

3. Matrix Multiplication

Know

- If A is $m \times n$ and B is $n \times p$, then AB exists and has dimensions $m \times p$.
- The inner dimensions must match; the outer dimensions give the dimensions of the product.
- Each entry of AB is found by multiplying a row of A by a column of B and adding the products.

- Matrix multiplication is generally not commutative: AB does not usually equal BA .
- AB can exist even when BA does not exist.

If A is 2×3 and B is 3×4 , then AB is 2×4 .

- Without multiplying, state whether each product is defined. If it is, state its dimensions.

A is 2×3 , B is 3×4 , C is 4×2 .

(a) AB (b) BA (c) BC (d) CA (e) AC

- Compute AB .

$$A = \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}$$

- Compute AB .

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 4 & -1 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \\ -2 & 6 \\ 0 & -3 \end{bmatrix}$$

- Compute both AB and BA . What does the result demonstrate?

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$$

- Find x and y from the matrix product.

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \end{bmatrix}$$

4. Identity, Zero Matrices, and Matrix Properties

Know

- The identity matrix I has 1s on its main diagonal and 0s elsewhere.
- For a compatible matrix A , $AI = IA = A$.
- A zero matrix O contains only zeros; $A + O = A$.
- Matrix addition is commutative and associative.
- Matrix multiplication is associative and distributive over addition, but generally not commutative.
- For products, the order of factors matters.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Verify by multiplication that $AI = A$.

$$A = \begin{bmatrix} 4 & -1 \\ 2 & 7 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- If A and B are 2×2 matrices, simplify the expression as far as matrix properties allow.

$$3A + 2B - A + 5B$$

- Is the statement always true, sometimes true, or never true? Explain.

$$AB = BA$$

- Give an example of two nonzero 2×2 matrices A and B for which AB is the zero matrix.

5. Powers of Matrices

Know

- Powers such as A^2 and A^3 are defined only for square matrices.
- A^2 means AA ; it does not mean squaring each entry.
- $A^0 = I$ for a square matrix A , just as a nonzero scalar raised to the zero power equals 1.
- Pattern recognition in repeated powers can make higher powers much faster to compute.

1. Compute A^2 .

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

2. Compute A^3 .

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

3. Without doing eight full multiplications, determine A^8 by first finding a pattern.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

4. Explain the difference between A^2 and squaring each entry of A .

6. Determinants

Know

- Only square matrices have determinants.
- For a 2×2 matrix, $\det(A) = ad - bc$.
- A square matrix is invertible exactly when its determinant is nonzero.
- For a 3×3 determinant, be comfortable expanding by cofactors if your course includes it.
- Swapping two rows changes the sign of the determinant; multiplying a row by k multiplies the determinant by k .

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

1. Find the determinant.

$$A = \begin{bmatrix} 5 & -2 \\ 3 & 7 \end{bmatrix}$$

2. Find all values of x for which the matrix is not invertible.

$$A = \begin{bmatrix} x & 4 \\ 3 & x-1 \end{bmatrix}$$

3. Compute the determinant of the 3×3 matrix by cofactor expansion.

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 4 \\ 5 & -2 & 1 \end{bmatrix}$$

4. Without expanding fully, determine how the determinant changes if the first row of a matrix is multiplied by -3 .

7. Inverse Matrices

Know

- The inverse A^{-1} satisfies $AA^{-1} = A^{-1}A = I$.
- A matrix has an inverse only if it is square and $\det(A)$ is not zero.
- For a 2×2 matrix, swap a and d , change the signs of b and c , then multiply by $1/(ad - bc)$.
- Always check the determinant before trying to find an inverse.

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

1. Find A^{-1} .

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$$

2. Determine whether the matrix is invertible. If it is, find its inverse.

$$B = \begin{bmatrix} 4 & 6 \\ -2 & -3 \end{bmatrix}$$

3. Find x so that the matrix has no inverse.

$$A = \begin{bmatrix} x & 6 \\ 2 & 3 \end{bmatrix}$$

4. Verify your inverse from Problem 1 by computing AA^{-1} .

8. Solving Matrix Equations

Know

- Matrix equations resemble ordinary algebra, but multiplication order must be preserved.
- If $AX = B$ and A is invertible, multiply on the LEFT by A^{-1} : $X = A^{-1}B$.
- If $XA = B$ and A is invertible, multiply on the RIGHT by A^{-1} : $X = BA^{-1}$.
- You cannot casually commute factors as you would with ordinary numbers.

1. Solve for X .

$$AX = B, \text{ where}$$

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

2. Solve for X .

$$XA = B, \text{ where}$$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$$

3. If P and Q are invertible square matrices, solve symbolically for X . Preserve the order of factors.

$$P X Q = R$$

4. Explain why multiplying $AX = B$ on the right by A^{-1} does not generally isolate X .

9. Systems of Equations Using Inverse Matrices

Know

- A linear system can be written as $AX = B$, where A is the coefficient matrix, X is the variable column, and B is the constants column.
- If A is invertible, the unique solution is $X = A^{-1}B$.
- If $\det(A) = 0$, the system does not have a unique solution; it may have no solution or infinitely many solutions.

$$\begin{array}{rcl} 2x + 3y = 7 & & \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \end{bmatrix} \\ 4x - y = 5 & \rightarrow & \begin{bmatrix} 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \end{bmatrix} \end{array}$$

- Write the system as $AX = B$ and solve using an inverse matrix.

$$\begin{array}{rcl} 3x + 2y & = & 11 \\ 5x - y & = & 9 \end{array}$$

- Solve using an inverse matrix.

$$\begin{array}{rcl} 2x - y & = & 1 \\ x + 3y & = & 14 \end{array}$$

- Explain what happens when you try to use the inverse-matrix method on this system.

$$\begin{array}{rcl} 2x + 4y & = & 8 \\ x + 2y & = & 5 \end{array}$$

- Write the following 3-variable system in matrix form $AX = B$. Do not solve.

$$\begin{array}{rcl} x + 2y - z & = & 4 \\ 3x - y + 2z & = & 10 \\ 2x + y + z & = & 7 \end{array}$$

10. Augmented Matrices and Elementary Row Operations

Know

- An augmented matrix records the coefficients and constants of a linear system.
- Three legal elementary row operations preserve the solution set: swap two rows, multiply a row by a nonzero constant, or add a multiple of one row to another row.
- Row operations should simplify the matrix toward echelon form or reduced row-echelon form.
- Be able to translate back and forth between a system of equations and its augmented matrix.

$$\begin{array}{rcl} x + 2y = 5 \\ 3x - y = 4 \end{array} \quad \text{becomes} \quad \begin{array}{c} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & -1 & 4 \end{array} \right] \end{array}$$

- Write the augmented matrix for the system.

$$\begin{array}{rcl} 2x - 3y + z & = & 7 \\ x + 4y - 2z & = & -1 \\ 5x + y + 3z & = & 12 \end{array}$$

- Write the system of equations represented by the augmented matrix.

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 3 & 1 & -4 \\ 2 & -1 & 0 & 7 \end{array} \right] \end{array}$$

- Perform the indicated row operation.

$$\begin{bmatrix} 1 & 2 & | & 5 \\ 3 & -1 & | & 4 \end{bmatrix}$$

$$R2 \rightarrow R2 - 3R1$$

4. Give a sequence of row operations that makes the first column below equal to $[1, 0, 0]^T$.

$$\begin{bmatrix} 2 & 1 & -1 & | & 3 \\ 4 & -2 & 3 & | & 8 \\ -2 & 5 & 1 & | & -1 \end{bmatrix}$$

11. Gaussian and Gauss-Jordan Elimination

Know

- Gaussian elimination reduces an augmented matrix to row-echelon form, then uses back-substitution.
- Gauss-Jordan elimination continues to reduced row-echelon form (RREF), where each pivot is 1 and is the only nonzero entry in its column.
- A row such as $[0 \ 0 \ 0 \ | \ c]$ with c nonzero signals an inconsistent system: no solution.
- A row of all zeros can indicate a free variable and potentially infinitely many solutions.
- For a hard test, be able to recognize unique, none, and infinitely-many-solution cases directly from an echelon matrix.

1. Solve by Gaussian or Gauss-Jordan elimination.

$$\begin{aligned} x + y + z &= 6 \\ 2x - y + 3z &= 14 \\ 3x + 2y - z &= 2 \end{aligned}$$

2. Solve using an augmented matrix.

$$\begin{aligned} 2x + 3y &= 13 \\ 4x - y &= 5 \end{aligned}$$

3. Determine whether the system has one solution, no solution, or infinitely many solutions. Explain from the matrix.

$$\begin{bmatrix} 1 & 2 & -1 & | & 4 \\ 0 & 1 & 3 & | & 2 \\ 0 & 0 & 0 & | & 5 \end{bmatrix}$$

4. Determine whether the system has one solution, no solution, or infinitely many solutions. Identify any free variables.

$$\begin{bmatrix} 1 & 0 & 2 & | & 3 \\ 0 & 1 & -4 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

12. Cramer's Rule

Know

- Cramer's Rule solves a square linear system using determinants when the coefficient determinant is nonzero.
- For a 2-variable system, $x = \det(A_x)/\det(A)$ and $y = \det(A_y)/\det(A)$.

- A_x is formed by replacing the x-column of A with the constants column; A_y replaces the y-column.
- Cramer's Rule is usually less efficient for large systems but is useful conceptually and on short problems.

1. Solve using Cramer's Rule.

$$\begin{aligned} 2x + 3y &= 7 \\ 5x - y &= 11 \end{aligned}$$

2. Solve using Cramer's Rule.

$$\begin{aligned} 4x - 2y &= 6 \\ 3x + 5y &= 19 \end{aligned}$$

3. Explain why Cramer's Rule cannot produce a unique solution when $\det(A) = 0$.

4. For the system below, set up A, A_x , and A_y and their determinants. Do not evaluate them.

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

13. Matrix Transformations and Applications

Know

- A 2 x 2 matrix can transform points in the coordinate plane by multiplying the matrix by a coordinate column vector.
- Common transformations include reflections, rotations, stretches/compressions, and shears.
- Matrices can also organize data, transition between states, or encode repeated processes.
- For applications, pay close attention to what rows and columns represent and to multiplication order.

Point (x, y) is written as the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$

1. Apply the transformation matrix to the point (3, -2).

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad v = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

2. Describe geometrically what the transformation does to points.

$$T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. A store sells notebooks and pens. Matrix P gives quantities purchased by two customers; matrix C gives unit prices. Compute PC and interpret each entry.

$$P = \begin{bmatrix} 3 & 5 \\ 2 & 8 \end{bmatrix} \quad C = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

4. A process is represented by the matrix below. If the initial state is v_0 , compute $v_1 = Av_0$ and then $v_2 = A^2v_0$.

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \quad v_0 = \begin{bmatrix} 100 \\ 50 \end{bmatrix}$$

14. Mixed Matrices Challenge

Directions

- These problems deliberately do not tell you which technique to use.
- Show enough work that you can check multiplication order, determinants, and row operations.

1. Find X.

$$\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} X = \begin{bmatrix} 5 & 1 \\ 4 & -2 \end{bmatrix}$$

2. Find all values of k for which the matrix is singular.

$$A = \begin{bmatrix} k & 2 \\ 6 & k-1 \end{bmatrix}$$

3. Compute $A^2 - 3A + 2I$.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

4. Solve the system using any matrix method.

$$\begin{aligned} x + 2y - z &= 3 \\ 2x - y + 3z &= 9 \\ 3x + y + 2z &= 10 \end{aligned}$$

5. Given $AB = I$ for square matrices A and B, what can you conclude about A and B? Explain.

6. A is 2×3 , B is 3×2 , and C is 2×2 . Determine which of the following are defined and give the dimensions: AB, BA, AC, CA, ABC, CBA.

7. Without computing an inverse, determine whether the system has a unique solution.

$$\begin{aligned} 4x + 6y &= 10 \\ 2x + 3y &= 7 \end{aligned}$$

8. A matrix transforms (1,0) to (2,1) and (0,1) to (-1,3). Write the 2×2 transformation matrix and use it to transform (4,-2).

9. A 3×3 augmented matrix reduces to the form below. Describe the solution set.

$$\begin{bmatrix} 1 & 0 & 4 & | & 7 \\ 0 & 1 & -2 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

10. For invertible matrices A and B, simplify $(AB)^{-1}$. Be careful about order.

Mastery Checklist

- ☐ I can identify matrix dimensions and individual entries.
- ☐ I can determine when matrices may be added, subtracted, or multiplied.
- ☐ I can add, subtract, and scalar-multiply matrices accurately.
- ☐ I can multiply matrices using row-by-column products.
- ☐ I remember that AB usually does not equal BA .
- ☐ I understand identity and zero matrices.
- ☐ I can compute powers of square matrices.

- ☐ I can calculate 2×2 determinants quickly.
- ☐ I can compute a 3×3 determinant by cofactor expansion if required.
- ☐ I can determine whether a matrix is invertible from its determinant.
- ☐ I can find the inverse of a 2×2 matrix.
- ☐ I can solve matrix equations while preserving multiplication order.
- ☐ I can write a linear system in the form $AX = B$.
- ☐ I can solve a system using an inverse matrix.
- ☐ I can write and interpret augmented matrices.
- ☐ I can perform elementary row operations correctly.
- ☐ I can solve systems by Gaussian or Gauss-Jordan elimination.
- ☐ I can recognize one solution, no solution, and infinitely many solutions from row-echelon form.
- ☐ I can use Cramer's Rule for a 2×2 system.
- ☐ I can apply a matrix to a coordinate vector and interpret simple transformations.
- ☐ I can solve mixed matrix problems without being told which technique to use.